Brill–Noether theory for Kuznetsov components

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Introduction

Question: Let X be a Fano threefold. Can $\mathrm{D}^b(X)$ determine X up to isomorphism?

Answer: Yes, by Bondal–Orlov's reconstruction theorem.

Question

Can "less" data than the whole derived category $\mathrm{D}^b(X)$ determine X up to isomorphism?

For some Fano threefolds, "less data" means the Kuznetsov component.

Example. For X a cubic threefold, an admissible subcategory of $\mathrm{D}^b(X)$ called the Kuznetsov component $\mathcal{K}u(X)$ determines it up to isomorphism [1].

In this poster, we focus on the case of X_{14} Fano threefolds.

X_{14} Fano threefolds

Definition. A X_{14} Fano threefold X is a Picard rank 1 ($\mathrm{Pic}(X) = \mathbb{Z}$), index 1 ($K_X = \mathcal{O}_X(-H)$), degree 14 ($K_X^3 = -14$) Fano ($-K_X$ is ample) threefold.

From now on, let X be a X_{14} Fano threefold, with H denoting its polarisation.

Definition. Let $\mathcal{E}$ be the restriction of the tautological sub-bundle on $\mathrm{Gr}(2,6)$. Define the **Kuznetsov component** of X to be

$$\begin{aligned}\mathcal{K}u(X) &:= \langle \mathcal{E}, \mathcal{O}_X\rangle^{\perp} \\ &= \{C \in \mathrm{D}^b(X) \mid \mathrm{Hom}^{\bullet}(D, C) = 0,\ D = \mathcal{E}, \mathcal{O}_X\}.\end{aligned}$$

It fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X\rangle$.

Bridgeland stability conditions

Weak stability conditions. A weak stability condition on $\mathrm{D}^b(X)$ is a pair $(Z, \mathcal{H})$ where $Z : K_0(\mathrm{D}^b(X)) \to \mathbb{C}$ is a weak stability function and $\mathcal{H}$ is a heart of $\mathrm{D}^b(X)$.

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Stability conditions on $\mathcal{K}u(X)$. To construct Bridgeland stability conditions on $\mathcal{K}u(X)$, take a tilt stability condition on $\mathrm{D}^b(X)$ (an example of a weak stability condition), tilt again and then restrict to $\mathcal{K}u(X)$.

Moduli spaces After fixing a Bridgeland stability condition σ on $\mathcal{A}_X$ and a class v in the numerical Grothendieck group of $\mathcal{A}_X$, one can consider moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, v)$ of σ-stable objects in $\mathcal{A}_X$.

$\mathcal{K}u(X)$ in the X_{14} case

By [2], there are birationally equivalent but non-isomorphic X_{14} Fano threefolds with equivalent Kuznetsov components, so the natural question in this case becomes:

Main question

What is the extra data along with $\mathcal{K}u(X)$ required to determine a X_{14} Fano threefold up to isomorphism?

The extra data

We claim that this **extra data** is a certain projection of the vector bundle $\mathcal{E}$ into $\mathcal{K}u(X)$.

Lemma

Consider the subcategory $\mathcal{D} := \langle \mathcal{K}u(X), \mathcal{E}\rangle \subset \mathrm{D}^b(X)$, the inclusion $i : \mathcal{K}u(X) \hookrightarrow \mathcal{D}$, and the right adjoint $i^!$ of i. We have

$$i^!(\mathcal{E}) = \mathrm{L}_{\mathcal{E}}\mathcal{Q}(-H)[1]$$

where $\mathcal{Q}$ is the restriction of the tautological quotient bundle on $\mathrm{Gr}(2,6)$. The object $i^!(\mathcal{E})$ is a two-term complex, and it is stable with respect to Bridgeland stability conditions on $\mathcal{K}u(X)$.

The idea

Recover X as a Brill–Noether (BN) locus inside

$$\mathcal{M} := \mathcal{M}_\sigma(\mathcal{K}u(X), [\mathrm{pr}(\mathcal{O}_x)[-1]])$$

where σ is a stability condition on $\mathcal{K}u(X)$ and $\mathrm{pr} = \mathrm{L}_{\mathcal{E}}\mathrm{L}_{\mathcal{O}_X} : \mathrm{D}^b(X) \to \mathcal{K}u(X)$ is the projection.

To do

- Show that $X \subset \mathcal{M}$
- Show that the points parametrising X are the only ones in $\mathcal{M}$ with a certain number of morphisms to a certain object, i.e. show X is a BN locus.

The **key point** is that this BN locus will be determined by $\mathcal{K}u(X)$ and the extra data $i^!(\mathcal{E})$, hence it will follow that X is determined precisely by $\mathcal{K}u(X)$ along with $i^!(\mathcal{E})$.

Showing that $X \subset \mathcal{M}$

The projection functor $\mathrm{pr} : \mathrm{D}^b(X) \to \mathcal{K}u(X)$ is Fourier–Mukai, so $\mathrm{pr} \cong \Phi_G$. We can define

$$\Phi_G \times \mathrm{id}_X := \Phi_{G \boxtimes \mathcal{O}_{\Delta_X}} : \mathrm{D}^b(X \times \mathcal{X}) \to \mathcal{K}u(X \times \mathcal{X}).$$

If $\mathcal{I}$ is the universal ideal sheaf on $X \times \mathcal{X}$, then $\Phi_{G \boxtimes \mathcal{O}_{\Delta_X}}(\mathcal{I})$ is a family of ideal sheaves on X parametrised by $\mathcal{X}$ which gives the **existence of a morphism**

$$p : \mathcal{X} \cong X \to \mathcal{M}.$$

Lemma

For $x \in X$, $p(x)$ is identified with $\mathrm{pr}(\mathcal{O}_x)[-1] \in \mathcal{M}$, and there is an embedding $p : X \hookrightarrow \mathcal{M}$ induced by pr.

Idea of proof. The key things to show are:

- The (shifted) projections of skyscrapers are σ-stable.
- $p(x) \neq p(y)$, i.e. $\mathrm{pr}(\mathcal{O}_x) \ncong \mathrm{pr}(\mathcal{O}_y)$ for $x \neq y$.
- The map on tangent spaces $\mathrm{d}p : T_x X \to T_{p(x)}\mathcal{M}$ induced by pr is well-defined and injective for all x.

Exhibiting X as a BN locus

Via mutation computations, one can show that there is a triangle

$$\mathcal{E}^{\oplus 4} \to I_x \to i^*(I_x) \to \mathcal{E}^{\oplus 4}[1]$$

where i^* is the right adjoint to $i : \mathcal{K}u(X) \hookrightarrow \mathcal{D}$.

Note that $i^*(I_x) = \mathrm{pr}(\mathcal{O}_X)[-1]$, and that $i^*(I_x)$ has four morphisms to $\mathcal{E}[1]$. Also note that

$$\mathrm{Hom}_{\mathcal{D}}(i^* I_x, \mathcal{E}[1]) \cong \mathrm{Hom}_{\mathcal{K}u(X)}(i^* I_x, i^! \mathcal{E}[1])$$

so it's equivalent to consider the number of morphisms to $i^!(\mathcal{E})[1]$.

Question: Are the $i^*(I_x)$'s the only objects in $\mathcal{M}$ which have 4 morphisms to $i^!(\mathcal{E})[1]$?

Definition. The **Brill–Noether locus** with respect to the extra data $i^!(\mathcal{E})$ is given by

$$\mathcal{BN} := \{F \in \mathcal{M} \mid \hom(F, i^!\mathcal{E}[1]) = 4\} \subset \mathcal{M}.$$

The answer to the above question is yes. In particular:

Theorem

We have $X \cong \mathcal{BN}$.

Idea of proof. Take $F \in \mathcal{BN} \setminus X$ and get a contradiction. The contradiction stems from the fact that F is the shift of a vector bundle.

Corollary

Let X and X' be X_{14} Fano threefolds, and let $i^!(\mathcal{E})$ and $i'^!(\mathcal{E}')$ be the associated extra data objects. Suppose that we have an equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ along with the isomorphism $\Phi(i^!(\mathcal{E})) \cong i'^!(\mathcal{E}')$. Then $X \cong X'$.

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