Categorical Torelli theorems for Gushel–Mukai threefolds

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Introduction

Question: Let X be a Fano threefold. Can $\mathrm{D}^b(X)$ determine X up to isomorphism?

Answer: Yes, by Bondal–Orlov's reconstruction theorem.

Question

Can "less" data than the whole derived category $\mathrm{D}^b(X)$ determine X up to isomorphism?

For some Fano threefolds, "less data" means the Kuznetsov component.

Example. For X a cubic threefold, an admissible subcategory of $\mathrm{D}^b(X)$ called the Kuznetsov component $\mathcal{K}u(X)$ determines it up to isomorphism [1].

In this poster, we focus on the case of **ordinary Gushel–Mukai (OGM)** threefolds.

GM threefolds

Definition. A GM threefold X is a Picard rank 1, index 1, degree 10 (genus 6) Fano threefold. There are two classes:

- X are quadric sections of a codimension 2 linear section of a Plücker embedded $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ (**OGM** threefolds) or;
- $X \xrightarrow{2:1} Y_5$ ramified in a quadric, where Y_5 is a codimension 3 linear section of a Plücker embedded $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ (special GM threefolds).

From now on, let X be an OGM threefold, with H denoting its polarisation.

Definition. Let $\mathcal{E}$ be the restriction of the tautological sub-bundle on $\mathrm{Gr}(2,5)$. Define the (alternative) **Kuznetsov component** of X to be

$$\begin{aligned}\mathcal{A}_X &:= \langle \mathcal{O}_X, \mathcal{E}^\vee \rangle^\perp \\ &= \{C \in \mathrm{D}^b(X) \mid \mathrm{Hom}^\bullet(D, C) = 0,\ D = \mathcal{O}_X, \mathcal{E}^\vee\}.\end{aligned}$$

It fits into the semiorthogonal decomposition $\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$.

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Bridgeland stability conditions

Weak stability conditions. A weak stability condition on $\mathrm{D}^b(X)$ is a pair $(Z, \mathcal{H})$ where $Z : K_0(\mathrm{D}^b(X)) \to \mathbb{C}$ is a weak stability function and $\mathcal{H}$ is a heart of $\mathrm{D}^b(X)$.

Stability conditions on $\mathcal{A}_X$. To construct Bridgeland stability conditions on $\mathcal{A}_X$, take a tilt stability condition on $\mathrm{D}^b(X)$ (an example of a weak stability condition), tilt again and then restrict to $\mathcal{A}_X$.

Moduli spaces After fixing a Bridgeland stability condition σ on $\mathcal{A}_X$ and a class v in the numerical Grothendieck group of $\mathcal{A}_X$, one can consider moduli spaces $\mathcal{M}_\sigma(\mathcal{A}_X, v)$ of σ-stable objects in $\mathcal{A}_X$.

Details can be found in [2].

$\mathcal{A}_X$ in the OGM case

By [3], there are birationally equivalent but non-isomorphic OGM threefolds with equivalent Kuznetsov components, so the natural questions in this case become:

Main questions

- Does $\mathcal{A}_X$ determine the birational equivalence class of X?
- What is the extra data along with $\mathcal{A}_X$ required to isolate an OGM from its birational equivalence class?

The extra data

We claim that this **extra data** is a certain projection of the vector bundle $\mathcal{E}$ into $\mathcal{K}u(X)$, an equivalent version of $\mathcal{A}_X$ (fitting in $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle$).

Lemma. ([4]) Consider the subcategory $\mathcal{D} := \langle \mathcal{K}u(X), \mathcal{E} \rangle \subset \mathrm{D}^b(X)$, the inclusion $i : \mathcal{K}u(X) \hookrightarrow \mathcal{D}$, and the right adjoint $\pi := i^!$ of i. We have

$$\pi(\mathcal{E}) = \mathrm{L}_{\mathcal{E}} \mathcal{Q}(-H)[1]$$

where $\mathcal{Q}$ is the restriction of the tautological quotient bundle on $\mathrm{Gr}(2,5)$. The object $\pi(\mathcal{E})$ is a two-term complex, and it is stable with respect to Bridgeland stability conditions on $\mathcal{K}u(X)$.

Refined categorical Torelli

Consider the projection $\mathrm{pr} := \mathrm{L}_{\mathcal{O}_X} \mathrm{L}_{\mathcal{E}^\vee} : \mathrm{D}^b(X) \to \mathcal{A}_X$, and let $C \subset X$ denote a conic.

Lemma. ([4])

- When $h^0(\mathcal{E}|_C) = 0$, we have $\mathrm{pr}(I_C) = I_C$.
- When $h^0(\mathcal{E}|_C) = 1$, the projection sits in the triangle

  $$\mathcal{E}[1] \to \mathrm{pr}(I_C) \to \mathcal{Q}^\vee,$$

  and there is a $\mathbb{P}^1$ of such conics in $\mathcal{C}(X)$, the Hilbert scheme of conics on X. Denote this $\mathbb{L}$.
- In both of the above cases $\mathrm{pr}(I_C)[1]$ is stable with respect to Bridgeland stability conditions on $\mathcal{A}_X$.

Remark. Under the standard equivalence $\Xi : \mathcal{K}u(X) \simeq \mathcal{A}_X$, we have the identification $\Xi(\pi(\mathcal{E})) \cong \mathrm{pr}(I_C)[1]$, in the case when $h^0(\mathcal{E}|_C) = 1$.

Theorem. ([4]) The functor pr produces an irreducible component $\mathcal{S} := p(\mathcal{C}(X))$ in $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$, where x is the numerical class of $\mathrm{pr}(I_C)$ and $p : \mathcal{C}(X) \to \mathcal{S}$ is a blow-down morphism contracting $\mathbb{L}$ to the smooth point associated to $\pi(\mathcal{E})$. In particular, $\mathcal{S}$ is isomorphic to the minimal model $\mathcal{C}_m(X)$ of $\mathcal{C}(X)$. Furthermore, the irreducible component $\mathcal{C}_m(X)$ is the whole moduli space $\mathcal{M}_\sigma(\mathcal{A}_X, -x)$.

Corollary

The data $(\mathcal{K}u(X), \pi(\mathcal{E}))$ determines X up to isomorphism ([4]).

Proof. By the theorem above,

$$\mathcal{M}_\sigma(\mathcal{A}_X, -x) \cong \mathcal{C}_m(X).$$

Blowing up $\mathcal{C}_m(X)$ at the point associated to $\pi(\mathcal{E})$ gives $\mathcal{C}(X)$, and by Logachev's reconstruction theorem X is recovered up to isomorphism from $\mathcal{C}(X)$.

Birational categorical Torelli

Theorem

Suppose we have an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then X is birational to X' ([4]).

Idea of proof. The other numerical (-1)-class of $\mathcal{A}_X$ is $y - 2x$, and one can show that

$$\mathcal{M}_\sigma(\mathcal{A}_X, y - 2x) \cong M_G^X(2,1,5)$$

where $M_G(2,1,5)$ is a Gieseker moduli space with the given Chern classes. So an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ sends $-x$ to either itself or $y - 2x$, and so gives rise to either $\mathcal{C}_m(X) \cong \mathcal{C}_m(X')$, or $\mathcal{C}_m(X) \cong M_G^{X'}(2,1,5)$. Results of [5] then imply that X is a certain birational transform of X' associated to either lines or conics.

Period maps for OGM threefolds

If $\mathcal{P}_{\mathrm{cat}} : \mathcal{X} \to \{\mathcal{K}u\}/\sim$ denotes a "**categorical period map**", then a corollary of the above theorem is that ([4])

$$\mathcal{P}_{\mathrm{cat}}^{-1}(\mathcal{A}_X) \cong \mathcal{C}_m(X)/\iota \cup M_G^X(2,1,5)/\iota'.$$

In [5], the authors (DIM) conjecture that the classical period map $\mathcal{P} : \mathcal{X} \to \mathcal{J}$ has fibers of the form $\mathcal{P}^{-1}(J(X)) = \mathcal{C}_m(X)/\iota \cup M/\iota'$ where $J(X)$ is the intermediate Jacobian and M is a surface birationally equivalent to $M_G^X(2,1,5)$. By our description of $\mathcal{P}_{\mathrm{cat}}^{-1}(\mathcal{A}_X)$, the DIM conjecture is equivalent to the following conjecture (in [4], we prove an **infinitesimal version** of this conjecture):

Conjecture

$J(X)$ determines $\mathcal{K}u(X)$.

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