CATEGORICAL TORELLI THEOREMS FOR HIGHER PICARD RANK FANO DOUBLE COVERS

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Abstract. We prove categorical Torelli theorems for three families of Fano threefolds with Picard rank $\rho > 1$. Members of these families manifest as double covers branched in anticanonical K3 surfaces. Our proof is based on reducing equivalences between the Kuznetsov components of Fano threefolds in the same deformation family to derived equivalences of their respective K3 branch divisors, and deducing that the resulting isomorphism of branch divisors gives rise to an isomorphism of the Fano threefolds for each family.

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1. Introduction

Let $\mathrm{D}^{\mathrm{b}}(X)$ be the bounded derived category of coherent sheaves on a variety X. The notion of a *subcategory* of $\mathrm{D}^{\mathrm{b}}(X)$ determining X is known as a *categorical Torelli theorem.* In the setting of smooth Fano threefolds of Picard rank 1, the question has been settled for almost all deformation families; see [PS23] for a survey. The subcategory in question in this context is known as the *Kuznetsov component* (see e.g. [Kuz14]).

In the present paper, we prove categorical Torelli theorems for three deformation families of Picard rank $\rho := \operatorname{rk} \operatorname{Pic} X > 1$ Fano threefolds over the field of the complex numbers $\mathbf{C}$. The members of the families are given as double covers $f\colon X \to Y$ branched in a divisor Z, where

1. **Family 2-6(b):** Y is a divisor of $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1,1)$ and Z is an anticanonical divisor. These are also known as *special Verra threefolds*;
2. **Family 2-8:** $Y = \mathrm{Bl}_p \mathbf{P}^3$ and Z is an anticanonical divisor, such that the intersection of Z and the exceptional divisor $\iota\colon E \to Y$ is smooth;
3. **Family 3-1:** $Y = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ and Z is a divisor of tridegree $(2,2,2)$.

The Kuznetsov components $\mathcal{A}_X$ of the three families can be defined as follows. The base Y of a Fano threefold X in Family 2-6(b) can be equivalently expressed as the projective bundle $Y \simeq \mathbf{P}(T_{\mathbf{P}^2})$, and the derived category of X has the following semiorthogonal decomposition (see (6.2))

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \pi^* \mathrm{D}^{\mathrm{b}}(\mathbf{P}^2) \rangle$$

where $\pi\colon \mathbf{P}(T_{\mathbf{P}^2}) \to \mathbf{P}^2$ is the projection.

The Fano threefolds of Family 2-8 have semiorthogonal decompositions (see Lemma 6.5)

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X(-H), \mathcal{O}_X(-E), \mathcal{O}_X \rangle,$$

where H and E are the class of the hyperplane and exceptional divisor on Y, respectively.

The Fano threefolds of Family 3-1 have semiorthogonal decompositions (see Lemma 6.11)

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X(0,0,0), \mathcal{O}_X(1,0,0), \mathcal{O}_X(0,1,0), \mathcal{O}_X(0,0,1) \rangle,$$

where e.g. $\mathcal{O}_X(1,0,0) = \mathcal{O}_{\mathbf{P}^1}(1) \boxtimes \mathcal{O}_{\mathbf{P}^1} \boxtimes \mathcal{O}_{\mathbf{P}^1}$. The subcategories $\mathcal{A}_X$ in each case are defined as the right orthogonals to the exceptional collections on their right hand sides, and these subcategories are known as the *Kuznetsov components* of each X, respectively. Our main theorem is as follows:

Theorem A (= Theorem 7.1)**.** *Let X and X' be very general Fano threefolds both of deformation type either 2-6(b), 2-8, or 3-1. Then an equivalence of Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ implies an isomorphism between X and X'.*

In the statement of the theorem later in the paper, we use a more precise notion of generality (see Definition 3.6).

Sketch of the proof. To prove Theorem A, we descend the equivalence of Kuznetsov components to an equivalence of μ_2-equivariant Kuznetsov components:

$$\mathcal{A}_X \simeq \mathcal{A}_{X'} \implies \mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2},$$

where μ_2 is generated by the canonical involution of the double cover. We show this by proving that in each case, we have an isomorphism of functors

$$S_{\mathcal{A}_X} \simeq \tau_{\mathcal{A}_X}[2],$$

where τ denotes the pullback along the covering involution, and $S_{\mathcal{A}_X}$ is the Serre functor of the Kuznetsov component (see Theorem 6.1). This result uses the techniques from [IK15].

In Theorem 6.2, we deduce that the equivariant Kuznetsov component is equivalent to the derived category of the branch divisor:

$$\mathcal{A}_X^{\mu_2} \simeq \mathrm{D^b}(Z), \quad \mathcal{A}_{X'}^{\mu_2} \simeq \mathrm{D^b}(Z')$$

We establish Theorem 6.2 with a study of several semiorthogonal decompositions of the equivariant derived category $\mathrm{D^b}(X)^{\mu_2}$ that arise from the geometry of X.

The next step is to study Fourier–Mukai partners of the branch divisors. Since the branch divisors are K3 surfaces, this can be done using the Derived Torelli Theorem for K3 surfaces [Muk87, Orl97]. The Derived Torelli Theorem was used in [HLOY04] and [MS24] to establish counting formulas for Fourier–Mukai partners of K3 surfaces. We use these counting formulas in Theorem 5.1 to show that the branch divisors have no non-trivial Fourier–Mukai partners:

$$\mathrm{D^b}(Z) \simeq \mathrm{D^b}(Z') \implies Z \simeq Z'.$$

The final piece of the proof is Theorem 4.1, which shows that the isomorphism class of the branch divisor Z determines X up to isomorphism:

$$Z \simeq Z' \implies X \simeq X'.$$

We prove this via an analysis of the geometry of the base Y of the double cover, combined with a study of the Neron–Severi lattices of the branch divisors. We put this all together in Section 7.

Relation to previous work. A similar approach was considered for special Gushel–Mukai threefolds in [JLLZ24, Theorem 9.9]. These are double covers of a linear section of the Grassmannian $\mathrm{Gr}(2,5)$ branched in a K3 surface. In the special Gushel–Mukai case, the equivariant Kuznetsov component is also equivalent to the derived category of the branch K3. However, since the Picard rank of a special Gushel–Mukai threefold is 1, the counting of Fourier–Mukai partners, as well as showing that the branch divisor determines the cover is much simpler. The deformation families of Fano threefolds (and in particular their branch divisors) considered in the present paper have Picard rank greater than 1, and the bases of two out of three of the families have non-rectangular Lefschetz decompositions. This necessitates a more subtle analysis.

Fano threefolds of Picard rank 1 manifesting as double covers with *canonically polarised* branch divisors were considered in [DJR25]. Due to the equivariant Kuznetsov component containing (rather than being equivalent to) the derived category of the branch divisor in that context, the analogous equivalences of equivariant Kuznetsov components in those cases were reduced to Hodge isometries of the middle primitive cohomologies of the branch divisors. Classical Torelli theorems were then employed to yield the desired categorical Torelli theorems.

Remark. There are two other deformation families of smooth Fano threefolds of Picard rank $\rho \geq 2$ which manifest as double covers that we do not consider, since the methods of the present paper do not apply to them (see Section 7.1.3).

Notation and conventions. We work over the field of complex numbers $\mathbf{C}$. All functors are assumed to be derived.

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2. Background

2.1. Derived categories.

For objects E, F in a triangulated category $\mathcal{C}$, we write

$$\mathrm{Hom}^\bullet(E,F) := \bigoplus_{i\in\mathbf{Z}} \mathrm{Ext}^i(E,F)[-i].$$

A subcategory $\mathcal{A} \subset \mathcal{C}$ is called *admissible* if both the left and right adjoint (denoted $i^!$ and i^*, respectively) to the inclusion $i\colon \mathcal{A} \hookrightarrow \mathcal{C}$ exist. An object $E \in \mathcal{C}$ is called *exceptional* if $\mathrm{Hom}^\bullet(E,E) = \mathbf{C}[0]$.

Definition 2.1. We say $\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a *semiorthogonal decomposition* of a triangulated category $\mathcal{C}$ if

(1) $\mathrm{Hom}^\bullet(F, G) = 0$ for all $F \in \mathcal{A}_i, G \in \mathcal{A}_j$ if $i > j$;
(2) for any $F \in \mathcal{C}$, there exists a sequence of morphisms

$$0 = F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 = F$$

such that $\mathrm{Cone}(F_i \to F_{i-1}) \in \mathcal{A}_i$.

When the subcategories $\mathcal{A}_i$ are all exceptional objects satisfying condition (1) above, we call $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ an *exceptional collection.* An exceptional collection in a category is called *full* if it generates the category.

Lemma 2.2. *If $\mathcal{C} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ is a semiorthogonal decomposition, then so are*

$$\mathcal{C} = \langle S_{\mathcal{C}}(\mathcal{A}_2), \mathcal{A}_1 \rangle = \langle \mathcal{A}_2, S_{\mathcal{C}}^{-1}(\mathcal{A}_1) \rangle,$$

where $S_{\mathcal{C}}$ is the Serre functor of $\mathcal{C}$.

Definition 2.3. Let $\mathcal{A} \subset \mathcal{C}$ be a subcategory. Define the *right orthogonal* $\mathcal{A}^{\perp}$ of $\mathcal{A}$ inside $\mathcal{C}$ to be the subcategory

$$\mathcal{A}^{\perp} \coloneqq \{F \in \mathcal{C} \mid \operatorname{Hom}^{\bullet}(G, F) = 0 \text{ for all } G \in \mathcal{A}\}$$

and the *left orthogonal* $\mathcal{A}^{\perp}$ of $\mathcal{A}$ inside $\mathcal{C}$ to be the subcategory

$$^{\perp}\mathcal{A} \coloneqq \{F \in \mathcal{C} \mid \operatorname{Hom}^{\bullet}(F, G) = 0 \text{ for all } G \in \mathcal{A}\}.$$

Definition 2.4. Let $i \colon \mathcal{A} \hookrightarrow \mathcal{C}$ be an admissible subcategory, and $F \in \mathcal{C}$ an object. We define the *left mutation of F along $\mathcal{A}$*, denoted $\mathbf{L}_{\mathcal{A}}F$, by the triangle

$$ii^{!}F \to F \to \mathbf{L}_{\mathcal{A}}F$$

and the *right mutation of F along $\mathcal{A}$*, denoted $\mathbf{R}_{\mathcal{A}}F$, by the triangle

$$\mathbf{R}_{\mathcal{A}}F \to F \to ii^{*}F.$$

In particular, when the admissible subcategory $\mathcal{A}$ is a single exceptional object E, the triangles defining the left and right mutations along E become

$$E \otimes \operatorname{Hom}^{\bullet}(E, F) \to F \to \mathbf{L}_E F$$

and

$$\mathbf{R}_E F \to F \to E \otimes \operatorname{Hom}^{\bullet}(F, E)^{\vee},$$

respectively.

Definition 2.5. Let X be a smooth projective variety, and let H be an ample divisor of X. The derived category $\mathrm{D^b}(X)$ is said to have a *rectangular Lefschetz decomposition* of length n if it admits a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(n) \rangle$$

where $\mathcal{A} \subset \mathcal{C}$ is an admissible subcategory, and $\mathcal{A}(i) \coloneqq \mathcal{A} \otimes \mathcal{O}_X(iH)$.

2.2. **Equivariant categories.** For this section, we mostly follow [KP17, Section 3].

Definition 2.6. Let $\mathcal{C}$ be a category, and G a finite group. An *action of G on $\mathcal{C}$* is the following data:

(1) For every $g \in G$, and autoequivalence $g^{*} \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$;
(2) For every $g, h \in G$, and isomorphism of functors $c_{g,h} \colon (gh)^{*} \xrightarrow{\sim} h^{*} \circ g^{*}$, such that the diagram

$$\begin{array}{ccc} (fgh)^{*} & \xrightarrow{c_{fg,h}} & h^{*} \circ (fg)^{*} \\ {\scriptstyle c_{f,gh}}\downarrow & & \downarrow{\scriptstyle h^{*}c_{f,g}} \\ (gh)^{*} \circ f^{*} & \xrightarrow{c_{g,h}f^{*}} & h^{*} \circ g^{*} \circ f^{*} \end{array}$$

commutes for all $f, g, h \in G$.

Definition 2.7. Let G be an action on $\mathcal{C}$. A *G-equivariant object* of $\mathcal{C}$ is a pair (F, ϕ), where F is an object of $\mathcal{C}$ and $\phi = \{\phi_g\}_{g \in G}$ is a collection of isomorphisms $\phi_g \colon F \to g^*F$ for $g \in G$, such that the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\phi_h} & h^*(F) & \xrightarrow{h^*(\phi_g)} & h^*(g^*(F)) \\ & \searrow^{\phi_{gh}} & & & \uparrow{\scriptstyle c_{g,h}(F)} \\ & & & & (gh)^*(F) \end{array}$$

commutes for all $g, h \in G$. The *G-equivariant category* $\mathcal{C}^G$ of $\mathcal{C}$ is the category of G-equivariant objects of $\mathcal{C}$.

If $\mathcal{C}$ is the derived category of a variety X, or if $\mathcal{C}$ is a semiorthogonal component of $\mathrm{D^b}(X)$, then $\mathcal{C}^G$ is also triangulated.

Theorem 2.8 ([Ela12, Theorem 6.3])**.** *Let X be a quasi-projective variety with an action of a finite group G. Assume that $\mathrm{D^b}(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a semiorthogonal decomposition preserved by G, i.e. each $\mathcal{A}_i$ is preserved by the action of G. Then there is a semiorthogonal decomposition*

$$\mathrm{D^b}(X)^G = \langle \mathcal{A}_1^G, \dots, \mathcal{A}_n^G \rangle.$$

An action of a group G on a category $\mathcal{C}$ is said to be *trivial* if for each $g \in G$, we have an isomorphism of functors $t_g \colon \mathrm{id} \xrightarrow{\sim} g^*$ such that

$$c_{g,h} \circ t_{gh} = h^* t_g \circ t_h$$

for all $g, h \in G$.

Proposition 2.9 ([KP17, Proposition 3.3])**.** *Let $\mathcal{C}$ be a triangulated category with a trivial action of a finite group G. Then $\mathcal{C}^G$ is also a triangulated category with a semiorthogonal decomposition*

$$\mathcal{C}^G = \langle \mathcal{C} \otimes \rho_0, \dots, \mathcal{C} \otimes \rho_n \rangle,$$

where $\rho_1, \dots, \rho_n$ are the finite-dimensional irreducible representations of G.

We will often abbreviate $\mathcal{C} \otimes \rho_i$ as $\mathcal{C}\rho_i$.

Definition 2.10. Let X and Y be schemes. A functor $\Phi_{\mathcal{E}} \colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ given by $\Phi_{\mathcal{E}}(F) \coloneqq p_{Y*}(p_X^*(F) \otimes \mathcal{E})$ is called a *Fourier–Mukai* functor with *Fourier–Mukai kernel* $\mathcal{E} \in \mathrm{D^b}(X \times Y)$, where p_X and p_Y are the obvious projections.

2.3. **Generalities on derived categories of cyclic covers.** Let $f \colon X \to Y$ be a double cover branched in $i \colon Z \hookrightarrow Y$. Denote the embedding of Z as the ramification divisor $j \colon Z \hookrightarrow X$. Let $\mu_2 = \{1, -1\}$ be the group generated by the involution associated to the double cover. Let $\widehat{\mu}_2 \simeq \mathbf{Z}/2\mathbf{Z}$ be the dual group of μ_2. Denote by $\rho_0, \rho_1 \in \widehat{\mu}_2$ the irreducible representations of μ_2, which correspond to the primitive characters of $\widehat{\mu}_2$. Let

$$f_k^* \colon \mathrm{D^b}(Y) \xrightarrow{-\otimes \rho_k} \mathrm{D^b}(Y)^{\mu_2} \xrightarrow{f^*} \mathrm{D^b}(X)^{\mu_2} \tag{2.1}$$

and

$$j_{k*} \colon \mathrm{D^b}(Z) \xrightarrow{-\otimes \rho_k} \mathrm{D^b}(Z)^{\mu_2} \xrightarrow{j_*} \mathrm{D^b}(X)^{\mu_2}. \tag{2.2}$$

Theorem 2.11 ([CP10, Theorem 5.1], [IU15, Theorem 1.6], [KP17, Theorem 4.1])**.** *For $k = 0, 1$, the functors f_k^* and j_{k*} are fully faithful. Moreover, we have a semiorthogonal decomposition*

$$\mathrm{D^b}(X)^{\mu_2} = \langle f_0^* \mathrm{D^b}(Y), j_{0*} \mathrm{D^b}(Z) \rangle.$$

Lemma 2.12 ([KP17, Lemma 5.1]). *Let Y be a smooth projective variety with a rectangular Lefschetz decomposition $\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(m-1) \rangle$. Let d be a positive integer such that $d < m$, and let $f\colon X \to Y$ be a double cover branched in a divisor in $|\mathcal{O}_Y(2d)|$.*

(1) The functor $f^*\colon \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(X)$ is fully faithful on the subcategory $\mathcal{B} \subset \mathrm{D}^{\mathrm{b}}(Y)$.

(2) Let $\mathcal{B}_X := f^*\mathcal{B}$. There is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \mathcal{B}_X(1), \dots, \mathcal{B}_X(m-d-1) \rangle$$

where the Kuznetsov component *$\mathcal{A}_X$ is the right orthogonal to*

$$\langle \mathcal{B}_X, \mathcal{B}_X(1), \dots, \mathcal{B}_X(m-d-1) \rangle.$$

When $\mathrm{D}^{\mathrm{b}}(Y)$ contains a (not necessarily full) exceptional collection and is not necessarily rectangular Lefschetz, we have the following.

Proposition 2.13 ([IK15, Lemma 3.1 and Proposition 3.2]). *Let $f\colon X \to Y$ be as above, and let Z be the branch divisor. Suppose for some $H \in \operatorname{Pic} Y$, we have $Z \sim 2H \sim -K_Y$. Assume $\underline{\mathcal{E}} = \{\mathcal{E}_1, \dots, \mathcal{E}_n\}$ is an exceptional collection in $\mathrm{D}^{\mathrm{b}}(Y)$ such that the collection*

$$\{\underline{\mathcal{E}}(-H), \underline{\mathcal{E}}\} \tag{2.3}$$

is also exceptional in $\mathrm{D}^{\mathrm{b}}(Y)$. Then the collection $f^*\underline{\mathcal{E}} \in \mathrm{D}^{\mathrm{b}}(X)$ is exceptional in $\mathrm{D}^{\mathrm{b}}(X)$, and in particular,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, f^*\underline{\mathcal{E}} \rangle$$

is a semiorthogonal decomposition, where the Kuznetsov component *$\mathcal{A}_X$ is the right orthogonal to the exceptional collection $f^*\underline{\mathcal{E}}$.*

Adding one extra assumption gives a convenient description of the Serre functor of the Kuznetsov component.

Proposition 2.14 ([IK15, Proposition 3.2]). *Suppose we are in the setting of Proposition 2.13, and further assume that the exceptional collection (2.3) is full in $\mathrm{D}^{\mathrm{b}}(Y)$. Let $\tau_{\mathcal{A}_X}$ be the involution functor induced by the involution τ of the double cover. Then the Serre functor of $\mathcal{A}_X$ is given by*

$$S_{\mathcal{A}_X} \simeq \tau_{\mathcal{A}_X}[\dim X - 1].$$

2.4. **Lattices.** In this subsection, we recall some facts about lattices necessary to study the Néron–Severi lattices of the branch divisors of our Fano threefolds. Our main reference for lattice theory is [Nik80].

A *lattice* is a free, finitely generated abelian group L equipped with a non-degenerate integral symmetric bilinear form $b\colon L \times L \to \mathbf{Z}$. For $v, u \in L$, we usually write $v \cdot u := b(v, u)$ and $v^2 := b(v, v)$. An isomorphism of lattices is called an *isometry.* The group of isometries of L is denoted $O(L)$. The *dual* of a lattice L is the group $L^* := \mathrm{Hom}(L, \mathbf{Z})$. The bilinear form on L induces an injective group homomorphism

$$\begin{aligned} L &\longrightarrow L^* \\ v &\longmapsto b(v, -). \end{aligned}$$

The cokernel $A_L := L^*/L$ is called the *discriminant group* of L, and the order of the discriminant group

$$\operatorname{disc}(L) := |A_L|$$

is called the *discriminant* of L. We say that L is *unimodular* if $\operatorname{disc}(L) = 1$. Since the dual can be characterised as

$$L^* \simeq \{x \in L \otimes \mathbf{Q} \mid \forall v \in L : x \cdot v \in \mathbf{Z}\},$$

it follows that L^* inherits a non-degenerate symmetric bilinear form taking values in $\mathbf{Q}$. If we assume our lattice L to be *even*, that is, we have $v^2 \in 2\mathbf{Z}$ for all $v \in L$, then A_L inherits a quadratic form $q\colon A_L \to \mathbf{Q}/2\mathbf{Z}$ from L^*. Note that any isometry $\sigma \in O(L)$ naturally induces an isometry $\overline{\sigma} \in O(A_L)$.

The *divisibility* of an element $v \in L$ is the positive integer

$$\operatorname{div}(v) \coloneqq \gcd_{u \in L}(v \cdot u). \tag{2.4}$$

For a lattice L, we define the *signature* $\operatorname{sgn}(L) = (r, s)$ of L to be the signature of the real quadratic space $L \otimes \mathbf{R}$. We say L is *indefinite* if $r, s > 0$.

Definition 2.15. The *genus* of a lattice L is the set, denoted $\mathcal{G}(L)$, of isomorphism classes of lattices L' that satisfy the following two conditions:

(1) We have $\operatorname{sgn}(L) = \operatorname{sgn}(L')$.
(2) There is an isomorphism $A_L \simeq A_{L'}$ respecting the natural quadratic forms on A_L and $A_{L'}$.

An embedding of lattices $N \hookrightarrow L$ is said to be *primitive* if the cokernel L/N is torsion-free. If, in addition, L is unimodular, we have two natural isomorphisms of groups:

$$\begin{array}{ccccc} & & \frac{L}{N \oplus N^\perp} & & \\ & \swarrow^{\simeq}_{f_N} & & \searrow^{\simeq}_{f_{N^\perp}} & \\ A_N & & & & A_{N^\perp}. \end{array}$$

Explicitly, an element $x \in L$ induces an element $(x \cdot -)|_N \in N^*$, and the homomorphism f_N sends the class of x in $\frac{L}{N \oplus N^\perp}$ to the class of $(x \cdot -)|_N$ in A_N. The group isomorphism $f_{N^\perp} \circ f_N^{-1}$ is an isometry if we multiply the quadratic form on $A_{N^\perp}$ by -1, i.e. we have an isometry

$$f_{N^\perp} \circ f_N^{-1}\colon A_N \simeq A_{N^\perp}(-1).$$

Lemma 2.16 ([Nik80, Proposition 1.6.1])**.** *Let $N \hookrightarrow L$ be a primitive embedding of even lattices, and suppose L is unimodular. Let $\sigma \in O(N)$. Then there exists an isometry $\tilde{\sigma} \in O(L)$ such that the diagram*

$$\begin{array}{ccc} N & \hookrightarrow & L \\ {\scriptstyle\sigma}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\tilde{\sigma}} \\ N & \hookrightarrow & L \end{array}$$

commutes, if and only if there is an isometry $\tau \in O(N^\perp)$ such that the diagram

$$\begin{array}{ccc} A_N & \xrightarrow{\simeq} & A_{N^\perp}(-1) \\ {\scriptstyle\overline{\sigma}}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\overline{\tau}} \\ A_N & \xrightarrow{\simeq} & A_{N^\perp}(-1) \end{array}$$

commutes.

An embedding of even lattices $N \hookrightarrow L$ is said to be an even *overlattice* if the cokernel L/N is a finite group. In this case, we have a sequence of finite-index embeddings

$$N \hookrightarrow L \hookrightarrow L^* \hookrightarrow N^*.$$

Therefore, the quotient $H_L \coloneqq L/N$ comes with a natural embedding into $A_N = N^*/N$.

Lemma 2.17 ([Nik80, Proposition 1.4.2]). *Let $N \hookrightarrow L$ and $N' \hookrightarrow L'$ be even overlattices. Then an isometry $\sigma\colon N \simeq N'$ can be extended to an isometry $\tilde{\sigma}\colon L \simeq L'$ if and only if*

$$\overline{\sigma}(H_L) = H_{L'}.$$

2.5. **Hodge structures and K3 surfaces.** We recall some fundamental facts about the integral cohomology groups of K3 surfaces. Our main reference for this subsection is [Huy16]. We assume all our K3 surfaces to be algebraic.

For a K3 surface Z, the group $H^2(Z, \mathbf{Z})$ is a free abelian group of rank 22. The cup-product on $H^2(Z, \mathbf{Z})$ is an integral, symmetric, non-degenerate, even bilinear form of signature $(3, 19)$, thus $H^2(Z, \mathbf{Z})$ is isometric to the *K3-lattice*:

$$\Lambda_{K3} \coloneqq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Here, U is the *hyperbolic plane*, which is the unique even unimodular lattice of signature $(1, 1)$, and E_8 is the unique even unimodular positive-definite lattice of rank 8.

Additionally, $H^2(Z, \mathbf{Z})$ carries a Hodge structure of weight 2:

$$H^2(Z, \mathbf{C}) \simeq H^{2,0}(Z) \oplus H^{1,1}(Z) \oplus H^{0,2}(Z),$$

and we have $H^{2,0}(Z) \simeq H^0(Z, K_Z) \simeq \mathbf{C}$. A group isomorphism $H^2(Z, \mathbf{Z}) \simeq H^2(Z', \mathbf{Z})$ is called a *Hodge isometry* if it is simultaneously an isometry and an isomorphism of Hodge structures. The *Néron–Severi lattice* of Z is $\mathrm{NS}(Z) \coloneqq H^{1,1}(Z) \cap H^2(Z, \mathbf{Z})$, seen as a sublattice of $H^2(Z, \mathbf{Z})$. Via the exponential sequence, we obtain an isomorphism $\operatorname{Pic} Z \simeq \mathrm{NS}(Z)$. The *transcendental lattice* $T(Z)$ of Z is defined to be the minimal, primitive, integral sub-Hodge structure of $H^2(Z, \mathbf{Z})$ such that $H^{2,0}(Z) \subset T(Z)_{\mathbf{C}}$. Since Z is algebraic, we have $T(Z) = \mathrm{NS}(Z)^\perp \subset H^2(Z, \mathbf{Z})$, see [Huy16, Lemma 3.3.1]. The transcendental lattice is a primitive sublattice of $H^2(Z, \mathbf{Z})$, and it also carries a natural Hodge structure inherited from $H^2(Z, \mathbf{Z})$.

Theorem 2.18 (Torelli Theorem for K3 surfaces, [Huy16, Theorem 7.5.3]). *Let Z and Z' be K3 surfaces, and let $\psi\colon H^2(Z, \mathbf{Z}) \simeq H^2(Z', \mathbf{Z})$ be a Hodge isometry. Then there exists an isomorphism $f\colon Z' \simeq Z$ such that $\psi = f^*$ if and only if ψ maps an ample class on Z to an ample class on Z'.*

Theorem 2.19 (Derived Torelli Theorem for K3 surfaces [Muk87, Orl97]). *Let Z and Z' be K3 surfaces. Then Z and Z' are derived equivalent if and only if there exists a Hodge isometry $T(Z) \simeq T(Z')$.*

Together, the Torelli Theorem and the Derived Torelli Theorem can be used to count Fourier–Mukai partners of K3 surfaces using the lattice theory of Nikulin [Nik80].

Definition 2.20. For a K3 surface Z, we denote by $\mathrm{FM}(Z)$ the set of isomorphism classes of K3 surfaces Z' such that $\mathrm{D^b}(Z) \simeq \mathrm{D^b}(Z')$. That is, $\mathrm{FM}(Z)$ is the set of Fourier–Mukai partners of Z.

Note that $\mathrm{FM}(Z)$ is never empty, as we have $Z \in \mathrm{FM}(Z)$.

Theorem 2.21 (Counting Formula for Fourier–Mukai partners of K3 surfaces, [HLOY04, Theorem 2.3]). *Let Z be a K3 surface. Then we have*

$$|\,\mathrm{FM}(Z)| = \sum_{N \in \mathcal{G}(\mathrm{NS}(Z))} |O(N)\backslash O(A_N)/O_{\mathrm{Hodge}}(T(Z))|.$$

Here, $\mathcal{G}(\mathrm{NS}(Z))$ is the genus of $\mathrm{NS}(Z)$, see Definition 2.15, and $O_{\mathrm{Hodge}}(T(Z))$ denotes the group of Hodge isometries of $T(Z)$.

From Theorem 2.21, we see that the number $|\operatorname{FM}(Z)|$ depends only on $\operatorname{NS}(Z)$ and $O_{\text{Hodge}}(T(Z))$, hence it is important to understand the group $O_{\text{Hodge}}(T(Z))$. Note that $\pm\mathrm{id}_{T(Z)}$ are always Hodge isometries, so that we have an injective group homomorphism $\mathbf{Z}/2\mathbf{Z} \hookrightarrow O_{\text{Hodge}}(T(Z))$. In fact, a K3 surface which is very general in a moduli space of lattice-polarised K3 surfaces of Picard rank $\rho \leq 19$ satisfies $O_{\text{Hodge}}(T(Z)) \simeq \mathbf{Z}/2\mathbf{Z}$, see for example [SZ20, Lemma 3.9]. Moreover, if ρ is odd, then we always have $O_{\text{Hodge}}(T(Z)) \simeq \mathbf{Z}/2\mathbf{Z}$:

Lemma 2.22 ([Ogu02, Lemma 4.1])**.** *Let Z be a K3 surface whose Picard rank ρ is odd. Then we have $O_{\text{Hodge}}(T(Z)) \simeq \mathbf{Z}/2\mathbf{Z}$.*

3. The families and their associated K3 surfaces

3.1. **Family 2-6(b).** The first family we study is Family 2-6(b) from Fanography [Bel26], sometimes also referred to as *special Verra threefolds*. We briefly recall the construction here. Consider a bidegree $(1,1)$-divisor $Y \subset \mathbf{P}^2 \times \mathbf{P}^2$ (in other words, Y is a member of Family 2-32 in [Bel26]). Let $Z \in |-K_Y|$ be an anticanonical divisor. Then there exists a branched double cover

$$X \xrightarrow{2:1} Y$$

with branch locus $Z \subset Y$. The Fano threefold X constructed in this way is a member of Family 2-6(b), and we will denote this deformation class by $\mathcal{X}_{2\text{-}6(b)}$. We will refer to Z as the K3 surface *associated to* X.

Remark 3.1. In this section, we study special Verra threefolds. On the other hand, Fano threefolds in Family 2-6(a) are called *ordinary Verra threefolds*. These are bidegree $(2,2)$ divisors in $\mathbf{P}^2 \times \mathbf{P}^2$, and will not be considered this section. A categorical Torelli theorem for ordinary Verra threefolds has already been sketched in [GRZ26, Remark 5.3].

The $(1,1)$-divisor Y comes equipped with two projections $p_i\colon Y \to \mathbf{P}^2$, and we denote

$$H_i := p_i^* L \in \operatorname{Pic} Y,$$

where $L \in \operatorname{Pic} \mathbf{P}^2$ is the class of a line. It follows from the Adjunction Formula that we have

$$K_Y = -2(H_1 + H_2).$$

Finally, let $i_Z\colon Z \hookrightarrow Y$ be the inclusion map. Then we write

$$h_i := i_Z^* H_i \in \operatorname{Pic} Z.$$

We now show that the classes h_1, h_2 form a basis for $\operatorname{Pic} Z$, under the assumption that Z is a very general member of $|-K_Y|$.

Proposition 3.2. *Let $Z \in |-K_Y|$.*

(1) If Z is a general element of $|-K_Y|$, then it is a smooth K3 surface;
(2) If Z is a very general element of $|-K_Y|$, then it is a smooth K3 surface and we have $\operatorname{Pic} Z = \operatorname{NS}(Z) = \mathbf{Z}[h_1] \oplus \mathbf{Z}[h_2]$ with Gram matrix

$$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}.$$

Proof. For (1), firstly, Z is smooth by Bertini's Theorem. By adjunction and since $Z \in |-K_Y|$, we have $K_Z = 0$. Furthermore, $H^1(Z, \mathcal{O}_Z) = 0$, hence Z is a K3 surface.

Note that we have $\operatorname{Pic} Y = \mathbf{Z}[H_1] \oplus \mathbf{Z}[H_2]$. Then $\operatorname{Pic} Z \simeq \operatorname{Pic} Y$ by [RS09, Theorem 1]. For the intersections, recall that as a homology class, $[Z] = 2H_1 + 2H_2$. Then $h_1^2 = H_1^2 \cdot [Z] = H_1^2 \cdot (2H_1 + 2H_2) = 2H_1^3 + 2H_1^2 \cdot H_2 = 2(1) + 0 = 2$. The rest of the intersections follow similarly. ∎

3.2. **Family 2-8.** Next, we consider Family 2-8. Let $p \in \mathbf{P}^3$ be a point, and denote the blow-up of $\mathbf{P}^3$ in p by $Y := \mathrm{Bl}_p\,\mathbf{P}^3$, and denote the blow-up morphism by

$$\pi\colon Y \to \mathbf{P}^3.$$

Let $Z \in |-K_Y|$ be an anticanonical divisor. Then there exists a branched double cover

$$X \xrightarrow{2:1} Y$$

with branch locus $Z \subset Y$. The Fano threefold X is a member of Family 2-8, and we will denote the deformation class of these Fano threefolds by $\mathcal{X}_{2\text{-}8}$.

We denote by $H \in \mathrm{Pic}\,Y$ the pullback of a plane in $\mathbf{P}^3$ along π, and by $E \in \mathrm{Pic}\,Y$ the exceptional divisor of π. Since Y is a blow-up, we have $\mathrm{Pic}\,Y = \mathbf{Z}[H] \oplus \mathbf{Z}[E]$. The canonical divisor of Y is

$$K_Y = 2E - 4H.$$

As before, let $i_Z\colon Z \hookrightarrow Y$ be the inclusion map. Then we denote

$$h := i_Z^* H, \quad e := i_Z^* E \in \mathrm{Pic}\,Z.$$

We now show that the classes h, e form a basis for $\mathrm{Pic}\,Z$, under the assumption that Z is a very general member of $|-K_Y|$.

Proposition 3.3. *Let $Z \in |-K_Y|$.*

(1) If Z is a general element of $|-K_Y|$, then it is a smooth K3 surface;
(2) If Z is a very general element of $|-K_Y|$, then it is a smooth K3 surface and we have $\mathrm{Pic}\,Z = \mathrm{NS}(Z) = \mathbf{Z}[h] \oplus \mathbf{Z}[e]$ where the Gram matrix for the lattice is

$$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Proof. For (1), the proof is the same as the proof of Proposition 3.2(1).

For (2), the fact that $\mathrm{Pic}\,Y \simeq \mathrm{Pic}\,Z$ follows from [BG12, Theorem 3.8]. For the lattice, firstly note that $h^2 = 4$ and $e^2 = -2$. Indeed, as a homology class, $[Z] = 4H - 2E$. Thus, $h^2 = H \cdot H \cdot [Z] = H \cdot H \cdot (4H - 2E) = 4H^3 - 2(H^2 \cdot E) = 4(1) - 2(0) = 4$. Also, $e^2 = E \cdot E \cdot [Z] = E \cdot E \cdot (4H - 2E) = 4(E^2 \cdot H) - 2E^3 = 4(0) - 2(1) = -2$. Furthermore, since e corresponds to the exceptional divisor, we have $h \cdot e = 0$. □

3.3. **Family 3-1.** Consider Family 3-1 from [Bel26]. That is, X is a double cover of

$$Y = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

branched in a divisor

$$Z \in |-K_Y| = |\mathcal{O}_Y(2,2,2)|.$$

For $1 \le i \le 3$, we write

$$H_i := p_i^* P,$$

where $p_i\colon Y \to \mathbf{P}^1$ denotes the projection onto the i-th factor, and $P \in \mathbf{P}^1$ is a closed point. In other words, we have for example $\mathcal{O}_Y(H_1) = \mathcal{O}_Y(1,0,0)$. Note that these three classes generate $\mathrm{Pic}\,Y$, i.e. we have

$$\mathrm{Pic}\,Y = \mathbf{Z}[H_1] \oplus \mathbf{Z}[H_2] \oplus \mathbf{Z}[H_3].$$

Finally, for $1 \le i \le 3$, let $h_i := H_i|_Z$ be the pullbacks of H_i to Z.

We show that the classes h_1, h_2, h_3 form a basis of $\mathrm{Pic}\,Z$, under the assumption that Z is a very general member of $|-K_Y|$.

Proposition 3.4. *Let $Z \in |-K_Y|$.*

(1) If Z is a general element of $|-K_Y|$, then it is a smooth K3 surface;

(2) *If Z is a very general element of $|-K_Y|$, then it is a smooth K3 surface and we have* $\operatorname{Pic} Z = \operatorname{NS}(Z) = \mathbf{Z}[h_1] \oplus \mathbf{Z}[h_2] \oplus \mathbf{Z}[h_3]$ *where the Gram matrix for the lattice is*

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Proof. For (1), the proof is again the same as the proof of Proposition 3.2(1).

For (2), the fact that $\operatorname{Pic} Y \simeq \operatorname{Pic} Z$ again follows from [BG12, Theorem 3.8]. For the lattice, firstly note that $H_i^2 = 0$ on each copy of $\mathbf{P}^1$, hence $h_i^2 = 0$ for $i = 1, 2, 3$. Now let $i \neq j$. As a homology class, $[Z] = 2H_1 + 2H_2 + 2H_3$. Then $h_i \cdot h_j = H_i \cdot H_j \cdot (2H_1 + 2H_2 + 2H_3) = 2$, since $H_i \cdot H_j \cdot H_k = 1$ for $k \in \{1, 2, 3\} \setminus \{i, j\}$, and the triple product vanishes for the other choices of k. □

3.4. **Z-generality.** From now on, whenever we consider a Fano threefold X contained in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$, we shall usually require the ramification locus to satisfy one of the conclusions of Proposition 3.2, Proposition 3.3, and Proposition 3.4, respectively.

Definition 3.5.

(1) We denote by $L_{2\text{-}6}$ the lattice of rank 2 with Gram matrix given by

$$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}. \tag{3.1}$$

(2) We denote by $L_{2\text{-}8}$ the lattice of rank 2 with Gram matrix given by

$$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

(3) We denote by $L_{3\text{-}1}$ the lattice of rank 2 with Gram matrix given by

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

As we saw in Proposition 3.2, Proposition 3.3, and Proposition 3.4, the Néron–Severi lattice of the branch divisor Z of a very general Fano threefold in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$ is isometric to $L_{2\text{-}6}$, $L_{2\text{-}8}$, and $L_{3\text{-}1}$, respectively.

Definition 3.6.

(1) A Fano threefold X in Family 2-6(b), Family 2-8, or Family 3-1 is called *K3-general* if the branch locus $Z \subset Y$ of the double cover $X \to Y$ is a smooth K3 surface.
(2) A Fano threefold in Family 2-6(b), Family 2-8, or Family 3-1 is called *Z-general* if it is K3-general and the Néron–Severi lattice of its associated K3 surface is isometric to $L_{2\text{-}6}$, $L_{2\text{-}8}$, or $L_{3\text{-}1}$, respectively.

Remark 3.7. By Proposition 3.2, Proposition 3.3, and Proposition 3.4, a general member of one of the three families under consideration is K3-general, and a very general member is Z-general.

Remark 3.8. Family 2-8 splits into two subfamilies called Family 2-8(a) and Family 2-8(b) on Fanography [Bel26]. We say that X is a member of Family 2-8(a) if the scheme-theoretic intersection $Z \cap E$ is smooth, and X is a member of Family 2-8(b) if $Z \cap E$ is singular but reduced. In this paper, we will mostly consider the former. More precisely, we prove a categorical Torelli theorem for Z-general Fano threefolds in Family 2-8. A Z-general Fano threefold in Family 2-8 is always a member of Family 2-8(a). Therefore, the categorical Torelli problem for very general members of Family 2-8(b) is still open.

4. The Fano is determined by the branch

In this section, we prove the following result.

Theorem 4.1. *Let X be a Z-general Fano threefold contained in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$, and let X' be a Fano threefold contained in the same family. Let Z and Z' be the branch loci of X and X', respectively. Then, if there is an isomorphism of the branch divisors, the Fano threefolds X and X' are isomorphic:*

$$Z \simeq Z' \implies X \simeq X'.$$

Proof. Combine Proposition 4.4, Proposition 4.13, and Proposition 4.16 below. □

Remark 4.2. Suppose X and X' are Fano threefolds in one of the three families under consideration. If X and X' have isomorphic branch loci, then X is K3-general (resp. Z-general) if and only if X' is. This follows immediately from the definition of K3-generality (resp. Z-generality), as it is a property of the branch divisor.

Remark 4.3. Theorem 4.1 can be equivalently expressed as follows: Let Y be a bidegree $(1, 1)$ divisor in $\mathbf{P}^2 \times \mathbf{P}^2$, the blow-up $\mathrm{Bl}_p(\mathbf{P}^3)$, or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Then the rational period map

$$|-K_Y|/\operatorname{Aut}(Y) \dashrightarrow \mathcal{M}_L$$

is generically injective, where $\mathcal{M}_L$ denotes the moduli space of L-polarised K3 surfaces. Here L is $L_{2\text{-}6}$, $L_{2\text{-}8}$, or $L_{3\text{-}1}$, respectively.

4.1. Family 2-6(b).

We first prove Theorem 4.1 for special Verra threefolds, i.e. for Family 2-6(b).

Proposition 4.4. *Let $X, X' \in \mathcal{X}_{2\text{-}6(b)}$ be Z-general. Then, if there is an isomorphism of branch divisors $Z \simeq Z'$, we have $X \simeq X'$.*

Before we prove Proposition 4.4, let us make some remarks on the geometry of the situation.

Definition 4.5.

(1) A *sextic structure* on a K3 surface Z is a double cover $f\colon Z \to \mathbf{P}^2$ branched in a sextic curve $C \subset \mathbf{P}^2$.
(2) If $f\colon Z \to \mathbf{P}^2$ is a sextic structure, then we call $h := f^*\mathcal{O}(1) \in \mathrm{NS}(Z)$ the *ample divisor of the sextic structure* f.
(3) Two sextic structures $f\colon Z \to \mathbf{P}^2$ and $f'\colon Z' \to \mathbf{P}^2$, with branch loci C and C', respectively, are said to be *isomorphic* if there is an automorphism $\phi\colon \mathbf{P}^2 \to \mathbf{P}^2$ satisfying $\phi(C) = C'$. Equivalently, there exists an isomorphism $Z \simeq Z'$ taking the ample divisor of f to the ample divisor of f'.
(4) The sextic structure on Z whose ample divisor is $h \in \mathrm{NS}(Z)$ is denoted f_h.

Recall that a Fano threefold in Family 2-6(b) is a double cover of a bidegree $(1,1)$ divisor Y in $\mathbf{P}^2 \times \mathbf{P}^2$. Let $Z \in |-K_Y|$ be a K3 surface, and denote by $p_i \colon Y \to \mathbf{P}^2$ the projection onto the i-th factor. Then the composition

$$Z \overset{i_Z}{\hookrightarrow} Y \overset{p_i}{\to} \mathbf{P}^2$$

is a sextic structure on Z, whose ample divisor is h_i, see Proposition 3.2. Here, i_Z denotes the inclusion map. Our first goal is to prove that Z admits precisely two sextic structures up to isomorphism, see Lemma 4.8 below.

Lemma 4.6. *The lattice $L_{2\text{-}6}$ contains no classes of square -2. As a consequence, for a K3 surface Z with $\mathrm{NS}(Z) \simeq L_{2\text{-}6}$, any class $D \in \mathrm{NS}(Z)$ which is effective and satisfies $D^2 > 0$ is ample, and for any class $h \in \mathrm{NS}(Z)$ with $h^2 > 0$, either h or $-h$ is effective.*

Proof. For the sake of contradiction, suppose that we have integers $a, b \in \mathbf{Z}$ such that

$$(ah_1 + bh_2)^2 = 2a^2 + 8ab + 2b^2 = -2. \tag{4.1}$$

Dividing (4.1) by 2 and reducing modulo 4, we find $a^2 + b^2 \equiv -1 \pmod 4$. However, the only squares modulo 4 are 0 and 1, so this is a contradiction. The final claim follows for example from [Huy16, Corollary 8.1.7]. □

Lemma 4.7. *Let Z be a K3 surface with $\mathrm{NS}(Z) \simeq L_{2\text{-}6}$. Let $h \in \mathrm{NS}(Z)$ be an effective class of square $h^2 = 2$, hence h is ample by Lemma 4.6. Then there exists another ample class $h' \in \mathrm{NS}(Z)$ such that the Gram matrix of $\mathrm{NS}(Z)$ with respect to the basis h, h' is*

$$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}.$$

In particular, there is an isometry $\sigma \in O(\mathrm{NS}(Z))$ such that $\sigma(h) = h_1$ and $\sigma(h') = h_2$.

Proof. Let $D \in \mathrm{NS}(Z)$ be any class such that h, D generate $\mathrm{NS}(Z)$. Such a class exists because h is primitive (as any class of square 2 must be primitive). We wish to find a $k \in \mathbf{Z}$ such that $(D + kh)^2 = 2$, that is,

$$\frac{1}{2}(D + kh)^2 = \frac{1}{2}D^2 + kD \cdot h + k^2 = 1. \tag{4.2}$$

This is a quadratic equation in k whose discriminant is $(D \cdot h)^2 - 2D^2 + 4 = 16$. Here, we use the fact that h, D generate $\mathrm{NS}(Z)$, hence the determinant of their Gram matrix must be $2D^2 - (D \cdot h)^2 = -12 = \mathrm{disc}(L_{2\text{-}6})$. In particular, there exist integer solutions to (4.2), hence we may find $h' = D + kh$ such that $(h')^2 = 2$. Possibly replacing h' by $-h'$, we may assume h' is effective by Lemma 4.6. This implies that $h \cdot h' \geq 0$, since $\mathrm{NS}(Z)$ does not contain any (-2)-curves. Since the determinant of the Gram matrix of the basis h, h' must be -12, we see $h \cdot h' = 4$. □

Lemma 4.8. *Let $Z \in |-K_Y|$ be a smooth K3 surface with $\mathrm{NS}(Z) = \langle h_1, h_2 \rangle \simeq L_{2\text{-}6}$. Then each sextic structure on Z is isomorphic to either f_{h_1} or f_{h_2}.*

Proof. Let $h \in \mathrm{NS}(Z)$ be an ample divisor with $h^2 = 2$, and let $h' \in \mathrm{NS}(Z)$ be the ample divisor obtained from Lemma 4.7. We claim that

$$\left\{ \frac{1}{2}h, \frac{1}{2}h' \right\} = \left\{ \frac{1}{2}h_1, \frac{1}{2}h_2 \right\} \subset A_{\mathrm{NS}(Z)}.$$

Indeed, it follows from a straightforward computation that we have

$$A_{\mathrm{NS}(Z)} = \left\langle \frac{1}{2}h_1 \right\rangle \oplus \left\langle \frac{1}{3}h_1 - \frac{1}{6}h_2 \right\rangle \simeq \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z}. \tag{4.3}$$

Hence $A_{\mathrm{NS}(Z)}$ contains precisely three elements of order 2, namely

$$\frac{1}{2}h_1, \quad \frac{1}{2}h_2, \quad \frac{1}{2}(h_1+h_2).$$

One easily checks that

$$q\left(\frac{1}{2}h_1\right) = q\left(\frac{1}{2}h_2\right) \equiv \frac{1}{2} \pmod{2\mathbf{Z}}, \quad q\left(\frac{1}{2}(h_1+h_2)\right) \equiv 1 \pmod{2\mathbf{Z}}.$$

Since $q(\frac{1}{2}h) \equiv \frac{1}{2} \pmod{2\mathbf{Z}}$, we must have $\frac{1}{2}h \in \{\frac{1}{2}h_1, \frac{1}{2}h_2\}$, as claimed. We assume $\frac{1}{2}h = \frac{1}{2}h_1$ and $\frac{1}{2}h' = \frac{1}{2}h_2$, as the other case follows by symmetry.

Let $\sigma\colon \mathrm{NS}(Z) \simeq \mathrm{NS}(Z)$ be an isometry such that $\sigma(h) = h_1$ and $\sigma(h') = h_2$. Such an isometry exists by Lemma 4.7. Note that σ preserves the ample cone, since h and h_1 are both ample. We claim that $\overline{\sigma} = \pm\mathrm{id}_{A_{\mathrm{NS}(Z)}}$. Indeed, $\overline{\sigma}$ fixes the two classes $\frac{1}{2}h_1$ and $\frac{1}{2}h_2$. Since $\overline{\sigma}$ fixes $\frac{1}{2}h_1$, we may write $\overline{\sigma} = (\sigma_1, \sigma_2)$, where $\sigma_1 = \mathrm{id}_{\langle \frac{1}{2}h_1 \rangle}$ and σ_2 is an isometry of $\left\langle \frac{1}{3}h_1 - \frac{1}{6}h_2 \right\rangle$. However, since $\left\langle \frac{1}{3}h_1 - \frac{1}{6}h_2 \right\rangle \simeq \mathbf{Z}/6\mathbf{Z}$, it follows that we must have $\sigma_2 = \pm\mathrm{id}$, so that $\overline{\sigma} = \pm\mathrm{id}_{A_{\mathrm{NS}(Z)}}$, as required. As a consequence, σ can be extended to a Hodge isometry $\Psi\colon H^2(Z,\mathbf{Z}) \simeq H^2(Z,\mathbf{Z})$ by Lemma 2.16. Since σ preserves the ample cone, $\Psi = \psi^*$ for some automorphism $\psi \in \mathrm{Aut}(Z)$, and this automorphism by construction satisfies $\psi^* h = h_1$, hence $f_h \simeq f_{h_1}$. □

Remark 4.9. As we can see in the proof of Lemma 4.8, a K3 surface Z with $\mathrm{NS}(Z) \simeq L_{2\text{-}6}$ has at most two sextic structures up to isomorphism. Moreover, the two sextic structures are isomorphic if and only if there exists a Hodge isometry $\Psi \in O_{\mathrm{Hodge}}(T(X))$ such that $\overline{\Psi} = \overline{\tau} \in O(A_{\mathrm{NS}(Z)})$, where τ is the isometry of $\mathrm{NS}(Z)$ which swaps h_1 and h_2. In the very general case, we have $O_{\mathrm{Hodge}}(T(Z)) \simeq \mathbf{Z}/2\mathbf{Z}$, and such a Hodge isometry does not exist, but in special cases there exist more Hodge isometries of $T(Z)$. Hence, there may exist Z-general Fano threefolds in Family 2-6(b) whose associated K3 surfaces have only one sextic structure up to isomorphism.

Lemma 4.10. *Let Y be a bidegree $(1,1)$ divisor in $\mathbf{P}^2 \times \mathbf{P}^2$, and let $p_i\colon Y \to \mathbf{P}^2$ denote the projection onto the i-th factor, where $i = 1, 2$. Then for each $i = 1, 2$ there is an isomorphism $Y \simeq \mathbf{P}(T_{\mathbf{P}^2})$ fitting into a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\ \simeq\ } & \mathbf{P}(T_{\mathbf{P}^2}) \\ & {\scriptstyle p_i}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathbf{P}^2 & \end{array}$$

where $\pi\colon \mathbf{P}(T_{\mathbf{P}^2}) \to \mathbf{P}^2$ denotes the canonical projection, and $T_{\mathbf{P}^2}$ is the tangent bundle of $\mathbf{P}^2$.

Proof. See [BF22, Lemma 5.7]. □

Recall that for an anticanonical K3 surface $Z \in |-K_Y|$, we write $i_Z\colon Z \hookrightarrow Y$ for the inclusion map.

Lemma 4.11. *Let $Y := \mathbf{P}(T_{\mathbf{P}^2})$, and let $Z, Z' \in |-K_Y|$ smooth K3 surfaces with $\mathrm{NS}(Z) \simeq L_{2\text{-}6} \simeq \mathrm{NS}(Z')$. Let $f := \pi \circ i_Z\colon Z \to \mathbf{P}^2$ and $g := \pi \circ i_{Z'}\colon Z' \to \mathbf{P}^2$ be the induced sextic structures on Z and Z'. Assume moreover that f and g are isomorphic via an automorphism $\phi\colon \mathbf{P}^2 \simeq \mathbf{P}^2$. Then ϕ lifts to an automorphism $\tilde{\phi}\colon Y \simeq Y$ such that $\tilde{\phi}(Z) = Z'$.*

Proof. By assumption, we have an isomorphism $\psi\colon Z \simeq Z'$ fitting into a commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow[\psi]{\simeq} & Z' \\ {\scriptstyle i_Z}\big\downarrow & & \big\downarrow{\scriptstyle i_{Z'}} \\ Y & & Y \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbf{P}^2 & \xrightarrow[\phi]{\simeq} & \mathbf{P}^2, \end{array}$$

where the compositions of the vertical maps are the sextic structures f and g. Since any automorphism of $\mathbf{P}^2$ lifts to an automorphism of Y, it follows that there exists an automorphism $\tilde{\phi} \in \operatorname{Aut}(Y)$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow[\tilde{\phi}]{\simeq} & Y \\ \big\downarrow & & \big\downarrow \\ \mathbf{P}^2 & \xrightarrow[\phi]{\simeq} & \mathbf{P}^2 \end{array}$$

commutes.

Therefore, $Z'' := (\tilde{\phi} \circ i_Z)(Z)$ and Z' are two K3 surfaces in Y such that the double covers $Z' \to \mathbf{P}^2$ and $Z'' \to \mathbf{P}^2$ have precisely the same branch locus $C \subset \mathbf{P}^2$. Finally, we may use Lemma 4.12 below, combined with Lemma 4.10, to conclude that $Z'' = Z'$, as required. ☐

Lemma 4.12. *Let Y be a bidegree $(1,1)$ divisor in $\mathbf{P}^2 \times \mathbf{P}^2$. Let $Z, Z' \in |-K_Y|$ be K3 surfaces with $\operatorname{NS}(Z) \simeq L_{2\text{-}6} \simeq \operatorname{NS}(Z')$. Let C and C' be the discriminant sextics of f_{h_1} and $f_{h'_1}$, respectively. Then, if $C = C'$ as subvarieties of $\mathbf{P}^2$, we have $Z = Z'$ as subvarieties of Y.*

Proof. Since $C = C'$, we find that Z and Z' are abstractly isomorphic, as they are both double covers of $\mathbf{P}^2$ with the same branch locus. Fix an isomorphism $\psi\colon Z \simeq Z'$ fitting in a commutative triangle

$$\begin{array}{ccc} Z & \xrightarrow[\psi]{\simeq} & Z' \\ & {\scriptstyle f_{h_1}}\searrow \quad \swarrow{\scriptstyle f_{h'_1}} & \\ & \mathbf{P}^2. & \end{array}$$

Then ψ also induces an isomorphism between f_{h_2} and $f_{h'_2}$:

$$\begin{array}{ccc} Z & \xrightarrow[\psi]{\simeq} & Z' \\ {\scriptstyle f_{h_2}}\big\downarrow & & \big\downarrow{\scriptstyle f_{h'_2}} \\ \mathbf{P}^2 & \xrightarrow[\phi]{\simeq} & \mathbf{P}^2. \end{array}$$

We claim that $\phi = \mathrm{id}_{\mathbf{P}^2}$. Note that this immediately implies the result, as the embedding $Z \hookrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ is the product of the two sextic structures f_{h_1} and f_{h_2}.

Since $(\mathrm{id}_{\mathbf{P}^2} \times \phi)(Z) = Z'$, we have

$$Z' \subset S := Y \cap \left((\mathrm{id}_{\mathbf{P}^2} \times \phi)(Y)\right). \tag{4.4}$$

Assume for the sake of contradiction that $\phi \neq \mathrm{id}_{\mathbf{P}^2}$. Then S is the intersection of Y with a different $(1,1)$ divisor in $\mathbf{P}^2 \times \mathbf{P}^2$, thus S is a divisor in Y. Moreover, we have $S = H_1 + H_2 \in \operatorname{Pic} Y$. Moreover, by (4.4), we can write $S = Z' + T$ for some effective divisor T. However, since $Z' \in |-K_Y|$, we have $Z' = 2H_1 + 2H_2$, thus $T = S - Z = -H_1 - H_2$.

Therefore T is not an effective divisor, which is a contradiction. Therefore, we have $\phi = \mathrm{id}_{\mathbf{P}^2}$, and the result follows. □

Proof of Proposition 4.4. By Lemma 4.8, there exists an isomorphism $\psi \colon Z \simeq Z'$, for which the ample divisor $\psi^* h_1'$ is equal to either h_1 or h_2. Suppose that we have $\psi^* h_1' = h_1$, as the other case follows by symmetry. Then, by Lemma 4.11, there is an automorphism $\tilde{\phi} \colon Y \simeq Y$ satisfying $\tilde{\phi}(Z) = Z'$, hence we have $X \simeq X'$. □

4.2. **Family 2-8.** In this subsection, we prove the following proposition.

Proposition 4.13. *Let $X, X' \in \mathcal{X}_{2\text{-}8}$ be Z-general. If there is an isomorphism of branch divisors $Z \simeq Z'$, then $X \simeq X'$.*

Before we proceed to the proof of Proposition 4.13, we first study the embeddings of the branch divisor Z into $Y = \mathrm{Bl}_p \mathbf{P}^3$.

Remark 4.14. The surface Z is the strict transform of a quartic hypersurface in $\mathbf{P}^3$ with an ordinary double point at p.

Indeed, let $\pi \colon Y \to \mathbf{P}^3$ be the blow-up map, and E the exceptional divisor. Denote $H_Y := \pi^* \mathcal{O}_{\mathbf{P}^3}(1)$. We have $K_Y = \pi^* K_{\mathbf{P}^3} + 2E = -4H + 2E$. Hence $-K_Y = 4H - 2E$. For such a blow-up, we have $\pi_* \mathcal{O}_Y(-kE) = I_p^k$ for $k \geq 0$. We have $\mathcal{O}_Y(-K_Y) = \mathcal{O}_Y(4H - 2E) = \pi^* \mathcal{O}_{\mathbf{P}^3}(4) \otimes \mathcal{O}_Y(-2E)$. Applying the projection formula, we get $\pi_* \mathcal{O}_Y(-K_Y) = \mathcal{O}_{\mathbf{P}^3}(4) \otimes \pi_* \mathcal{O}_Y(-2E) = I_p^2(4)$. This means that the global sections of $\mathcal{O}_Y(-K_Y)$ are precisely degree-4 polynomials on $\mathbf{P}^3$ vanishing to order at least 2 at the point p, in other words, we have a double point at p. Taking the strict transform removes the multiplicity-2 component along E, giving a divisor in $|-K_Y|$.

Lemma 4.15. *Let $Z \in |-K_Y|$ be a general anticanonical divisor of $Y = \mathrm{Bl}_p \mathbf{P}^3$, so that Z is a smooth K3 surface. Let $i \colon Z \hookrightarrow Y$ and $j \colon Z \hookrightarrow Y$ be two embeddings. Assume that $i^*(H) = j^*(H) = h \in \mathrm{NS}(Z)$. Then there is an automorphism $\psi \colon Y \simeq Y$ making the following diagram commute:*

$$\begin{array}{ccccc} & & Z & & \\ & \overset{i}{\swarrow} & & \overset{j}{\searrow} & \\ Y & & \xrightarrow[\psi]{\simeq} & & Y \end{array} \tag{4.5}$$

Proof. Consider the morphism $\phi := \phi_{|h|} \colon Z \to \mathbf{P}^3$. The image $\phi(Z)$ is a singular quartic K3 surface with an ordinary double point in p. Explicitly, the map ϕ is the restriction of the blow-up $\mathrm{Bl}_p \colon Y \to \mathbf{P}^3$, and the exceptional (-2)-curve of Z is contracted to an ordinary double point (see Remark 4.14).

By the assumption that $i^* H = j^* H = h$, there exists an automorphism $\psi' \colon \mathbf{P}^3 \simeq \mathbf{P}^3$ such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \overset{i}{\swarrow} & & \overset{j}{\searrow} & \\ Y & & & & Y \\ \downarrow & & & & \downarrow \\ \mathbf{P}^3 & & \xrightarrow[\psi']{\simeq} & & \mathbf{P}^3 \end{array}$$

commutes. In particular, ψ' fixes the point $p \in \mathbf{P}^3$, hence it lifts to the desired automorphism $\psi \in \mathrm{Aut}(Y)$ making (4.5) commute. □

Proof of Proposition 4.13. By Z-generality of X, the Néron–Severi lattice $\mathrm{NS}(Z) = \mathrm{Pic}\, Z$ is given by the Gram matrix

$$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Suppose we have an isomorphism of branch divisors $\phi\colon Z \simeq Z'$. In this setting, $\mathrm{NS}(Z)$ is generated by $h := H|_Z$ and $e := E|_Z$, and $\mathrm{NS}(Z')$ is similarly generated by h' and e' by Proposition 3.3. The isomorphism ϕ induces an isometry $\phi^*\colon \mathrm{NS}(Z) \simeq \mathrm{NS}(Z')$. We claim we must have $\phi^*(h) = h'$ and $\phi^*(e) = e'$.

Indeed, let $a, b \in \mathbf{Z}$ such that $\phi^*(h) = ah' + be'$. Since ϕ is an isomorphism, $\phi^*(h)$ must be effective, since h is effective. It follows that $\phi(h) \cdot e' = -2b^2 \geq 0$, hence $b = 0$. From this, it easily follows that $a = \pm 1$. However, since $-h'$ is not effective, we obtain $\phi(h) = h'$. Now, since e is orthogonal to h, it follows that $\phi(e)$ is orthogonal to $\phi(h) = h'$. We have $h'^{\perp} = \langle e' \rangle \subset \mathrm{NS}(Z')$, so we obtain $\phi(e) = \pm e'$. Finally, since e is effective, $\phi(e)$ is also effective, and we must have $\phi(e) = e'$.

Now, write $i_Z\colon Z \hookrightarrow Y$ and $i_{Z'}\colon Z' \hookrightarrow Y$ for the inclusions. Then i_Z and the composition $i_{Z'} \circ \phi$ are two different embeddings of Z into Y, and by the above computations we have

$$i_Z^*(H) = h = \phi^*(h') = (i_{Z'} \circ \phi)^*(H).$$

This means that we may use Lemma 4.15 to obtain an automorphism $\psi \in \mathrm{Aut}(Y)$ fitting into a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow[\phi]{\simeq} & Z' \\ {\scriptstyle i_Z}\big\downarrow & & \big\downarrow{\scriptstyle i_{Z'}} \\ Y & \xrightarrow[\psi]{\simeq} & Y. \end{array}$$

From this, we immediately obtain an isomorphism of the double covers $X \simeq X'$, as required. □

4.3. **Family 3-1.** Finally, we consider Theorem 4.1 for Family 3-1.

Proposition 4.16. *Let $X, X' \in \mathcal{X}_{3\text{-}1}$ be Z-general. If there is an isomorphism of branch divisors $Z \simeq Z'$, then $X \simeq X'$.*

Definition 4.17.

(1) An *elliptic fibration* of a K3 surface Z is a surjective morphism $f\colon Z \to \mathbf{P}^1$.
(2) Let $f\colon Z \to \mathbf{P}^1$ be an elliptic fibration, and let $x \in \mathbf{P}^1$ be a closed point. Then the class $[Z_x] \in \mathrm{NS}(Z)$ is called the *fibre class* of f. The fibre class is independent of the chosen point x.
(3) For an elliptic fibration $f\colon Z \to \mathbf{P}^1$ with fibre class $F \in \mathrm{NS}(Z)$, a curve $C \subset Z$ with the property that $C \cdot F > 0$ is called a *multisection* of f.
(4) The *multisection index* $t \geq 1$ of an elliptic fibration $f\colon Z \to \mathbf{P}^1$ is the minimal degree of a multisection. If $F \in \mathrm{NS}(Z)$ is the fibre class of f, then t is the divisibility of F in $\mathrm{NS}(Z)$ (see (2.4)):
$$t = \mathrm{div}(F).$$
(5) Two elliptic fibrations $f\colon Z \to \mathbf{P}^1$, $f'\colon Z' \to \mathbf{P}^1$ are said to be *isomorphic* if there is a commutative square as below, where the horizontal maps are isomorphisms

$$\begin{array}{ccc} Z & \xrightarrow{\simeq} & Z' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ \mathbf{P}^1 & \xrightarrow{\simeq} & \mathbf{P}^1. \end{array}$$

Equivalently, there is an isomorphism $\phi\colon Z \simeq Z'$ satisfying $\phi^* F' = F$, where F and F' are the fibre classes of f and f', respectively.

Recall that a Fano threefold in Family 3-1 is a double cover of $Y = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ branched in an anticanonical K3 surface $Z \subset Y$. For such a K3 surface, the three projections $Z \to \mathbf{P}^1$ are elliptic fibrations of multisection index 2 with fibre classes given by h_1, h_2 and h_3, respectively. Our first goal is to show that Z admits at most three elliptic fibrations up to isomorphism, provided $\mathrm{NS}(Z) \simeq L_{2\text{-}6}$, see Lemma 4.20 below.

Lemma 4.18. *Let $F \in L_{3\text{-}1}$ be a primitive isotropic class. Then there exist primitive isotropic classes F', F'' such that F, F', F'' is a basis of $\mathrm{NS}(Z)$ whose Gram matrix is*

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

In particular, for each $1 \leq i \leq 3$, there exists an isometry $\sigma \in O(L_{3\text{-}1})$ such that $\sigma(F) = h_i$.

Proof. Firstly we note that F has divisibility 2 (indeed, every primitive class in $L_{3\text{-}1}$ has divisibility 2). Therefore, we may find a primitive class D' such that $F \cdot D' = 2$. Note that $D'^2 = 4n$ for some $n \in \mathbf{Z}$, thus $(D' - nF)^2 = 0$. Thus, denoting $F' := D' - nF$, we now have a primitive sublattice $\langle F, F' \rangle \subset L_{3\text{-}1}$ with Gram matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

We may now choose a second primitive class $D'' \in L_{3\text{-}1}$ such that F, F', D'' is a $\mathbf{Z}$-basis for $L_{3\text{-}1}$. Its Gram matrix is given by

$$M := \begin{pmatrix} 0 & 2 & 2a \\ 2 & 0 & 2b \\ 2a & 2b & 4c \end{pmatrix}$$

for some $a, b, c \in \mathbf{Z}$ such that $\det M = 16ab - 16c = 16$, hence $c = ab - 1$. Now note that

$$F'' := (1-a)F + (1-b)F' + D''$$

satisfies $F \cdot F'' = F' \cdot F'' = 2$ and $F''^2 = 0$, and F, F', F'' is a $\mathbf{Z}$-basis of $L_{3\text{-}1}$. Therefore, the group automorphism $\sigma \in \mathrm{Aut}(L_{3\text{-}1})$ defined by $\sigma(F) = h_1$, $\sigma(F') = h_2$ and $\sigma(F'') = h_3$ is an isometry. This finishes the proof, since there is a natural action of the symmetric group S_3 on $L_{3\text{-}1}$ by interchanging the basis vectors of $L_{3\text{-}1}$. □

Next,we prove a technical lemma about isotropic subgroups of the discriminant group $A_{3\text{-}1} := A_{L_{3\text{-}1}}$. Note that the order of the discriminant group is $|A_{3\text{-}1}| = \det(L_{3\text{-}1}) = 16$. This lemma will also be relevant in Section 5, where we study Fourier–Mukai partners of the branch divisors.

Lemma 4.19. *The discriminant group $A_{3\text{-}1}$ contains precisely 3 isotropic subgroups of order 2, and does not contain any isotropic subgroups of order 4. The three isotropic subgroups of order 2 are those generated by $\frac{1}{2}h_1$, $\frac{1}{2}h_2$, and $\frac{1}{2}h_3$, respectively.*

Proof. The discriminant group $A_{3\text{-}1}$ is generated by the elements

$$\begin{aligned} h_1^* &= \tfrac{1}{4}(-h_1 + h_2 + h_3) \\ h_2^* &= \tfrac{1}{4}(h_1 - h_2 + h_3) \\ h_3^* &= \tfrac{1}{4}(h_1 + h_2 - h_3). \end{aligned}$$

However, these three elements are not independent in $A_{3\text{-}1}$, and a more convenient generating set of $A_{3\text{-}1}$ is

$$\frac{1}{2}h_1, \qquad \frac{1}{2}h_2, \qquad \frac{1}{4}(h_1+h_2-h_3).$$

Indeed, we have

$$A_{3\text{-}1} = \left\langle \frac{1}{2}h_1 \right\rangle \oplus \left\langle \frac{1}{2}h_2 \right\rangle \oplus \left\langle \frac{1}{4}(h_1+h_2-h_3) \right\rangle \simeq \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}. \tag{4.6}$$

Since h_1, h_2 and h_3 are isotropic, the subgroups

$$\left\langle \frac{1}{2}h_1 \right\rangle, \quad \left\langle \frac{1}{2}h_2 \right\rangle, \quad \left\langle \frac{1}{2}h_3 \right\rangle \tag{4.7}$$

are isotropic subgroups of order 2 in $A_{3\text{-}1}$. Moreover, using (4.6), we see that $A_{3\text{-}1}$ has precisely 6 subgroups of order 2, and it is not difficult to show that precisely three of them are isotropic, namely those appearing in (4.7). Finally, a straightforward computation shows that $A_{3\text{-}1}$ has no isotropic subgroups of order 4. □

We are now ready to prove that a K3 surface Z with $\mathrm{NS}(Z) \simeq L_{3\text{-}1}$ has at most three elliptic fibrations up to isomorphism.

Lemma 4.20. *Let Z be a K3 surface with $\mathrm{NS}(Z) \simeq L_{3\text{-}1}$. Suppose $Z \to \mathbf{P}^1$ is an elliptic fibration with fibre class $F \in \mathrm{NS}(Z)$. Then there is an isomorphism $\phi\colon Z \simeq Z$ such that $\phi^* F = h_i$ for some $1 \leq i \leq 3$.*

Proof. By Lemma 4.18, we may find isotropic classes $F', F'' \in \mathrm{NS}(Z)$ such that F, F', F'' generate $\mathrm{NS}(Z)$, and such that their Gram matrix is precisely that of $L_{3\text{-}1}$. It follows that $\frac{1}{2}F, \frac{1}{2}F', \frac{1}{2}F''$ generate three different isotropic subgroups of order 2 in $A_{3\text{-}1}$. By Lemma 4.19, these three subgroups are the subgroups generated by $\frac{1}{2}h_i$ for $1 \leq i \leq 3$. We will assume

$$\left\langle \frac{1}{2}F \right\rangle = \left\langle \frac{1}{2}h_1 \right\rangle, \quad \left\langle \frac{1}{2}F' \right\rangle = \left\langle \frac{1}{2}h_2 \right\rangle, \quad \left\langle \frac{1}{2}F'' \right\rangle = \left\langle \frac{1}{2}h_3 \right\rangle$$

as the other 5 cases follow by symmetry. Now let $\sigma \in O(\mathrm{NS}(Z))$ be the isometry with $\sigma(F) = h_1$, $\sigma(F') = h_2$ and $\sigma(F'') = h_3$. Then σ preserves the ample cone, and using (4.6) one sees that $\overline{\sigma} = \pm \mathrm{id}_{A_{\mathrm{NS}(Z)}}$, hence $\sigma = \phi^*$ for some automorphism $\phi \in \mathrm{Aut}(Z)$, which finishes the proof. □

Remark 4.21. Similarly to Remark 4.9, a K3 surface Z with $\mathrm{NS}(Z) \simeq L_{3\text{-}1}$ has precisely three elliptic fibrations up to isomorphism if $O_{\mathrm{Hodge}}(T(Z)) \simeq \mathbf{Z}/2\mathbf{Z}$, but this number may drop for special K3 surfaces which admit more Hodge isometries of $T(Z)$.

As an immediate consequence of Lemma 4.20, we obtain the following result.

Lemma 4.22. *Let $Z, Z' \in |-K_Y|$ be K3 surfaces with $\mathrm{NS}(Z) \simeq L_{3\text{-}1} \simeq \mathrm{NS}(Z')$. Suppose we have $Z \simeq Z'$. Then there exists an isomorphism $\phi\colon Z \simeq Z'$ and a permutation $\sigma \in S_3$ such that $\phi^*(h_i') = h_{\sigma(i)}$ for all $1 \leq i \leq 3$.*

Let Z be the branch divisor of a Z-general Fano threefold X in the family $\mathcal{X}_{3\text{-}1}$. Then the three effective, primitive, isotropic classes $h_i \in \mathrm{NS}(Z)$ induce three elliptic fibrations $f_i\colon Z \to \mathbf{P}^1$ of multisection index 2 (see Definition 4.17). We note that the inclusion $Z \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ is simply the product of these three elliptic fibrations.

Proof of Proposition 4.16. Let $\phi\colon Z \simeq Z'$ be an isomorphism. By Lemma 4.22, it follows that there exists a permutation $\sigma \in S_3$ such that $\phi^*(h'_i) = h_{\sigma(i)}$ for all $1 \leq i \leq 3$. In particular, for each $1 \leq i \leq 3$, there is a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow[\phi]{\simeq} & Z' \\ {\scriptstyle f_{\sigma(i)}}\downarrow & & \downarrow{\scriptstyle f'_i} \\ \mathbf{P}^1 & \xrightarrow[\psi_i]{\simeq} & \mathbf{P}^1. \end{array}$$

Denote

$$\begin{array}{rccc} \phi_\sigma: & \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \\ & (x_1, x_2, x_3) & \longmapsto & (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \end{array}$$

and

$$\psi \coloneqq (\psi_1 \times \psi_2 \times \psi_3) \circ \phi_\sigma.$$

Then we see that there is a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow[\phi]{\simeq} & Z' \\ {\scriptstyle i_Z}\big\downarrow & & \big\downarrow{\scriptstyle i_{Z'}} \\ \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 & \xrightarrow[\psi]{\simeq} & \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1, \end{array}$$

which shows that we have an isomorphism $X \simeq X'$. □

5. Fourier–Mukai partners of the associated K3 surfaces

The main goal of this section is to prove the following result.

Theorem 5.1. *Let X be a Z-general Fano variety in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$. Let Z be the associated K3 surface of X. Then Z has no non-trivial Fourier–Mukai partners. That is, for any K3 surface Z', we have*

$$\mathrm{D^b}(Z) \simeq \mathrm{D^b}(Z') \implies Z \simeq Z'.$$

Proof. Combine Proposition 5.6 and Proposition 5.7 below. □

Recall that X is said to be Z-*general* if the Néron–Severi lattice of Z is isometric to $L_{2\text{-}6}$, $L_{2\text{-}8}$, or $L_{3\text{-}1}$, see Definition 3.6.

5.1. Families 2-6(b) and 2-8.

In this subsection, we prove Theorem 5.1 for the families 2-6(b) and 2-8. We use the Counting Formula for Fourier–Mukai partners of K3 surfaces of [HLOY04], see Theorem 2.21.

We first show that $L_{2\text{-}6}$ and $L_{2\text{-}8}$ are each unique in their genus. This result was also verified using the OSCAR function `genus_representatives`, developed by Simon Brandhorst and Stevell Muller [OSC26, DEF+25].

To compute the genera of $L_{2\text{-}6}$ and $L_{2\text{-}8}$, we use the well-known correspondence between lattices and quadratic forms. Our main reference for binary quadratic forms is [CS99, Chapter 15]. Explicitly, the indefinite even lattice

$$L = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

corresponds to the binary quadratic form $f_L \coloneqq ax^2 + bxy + cy^2$. The discriminant of f_L is defined to be $\mathrm{disc}(f_L) \coloneqq b^2 - 4ac$ (note that this coincides with our definition of $\mathrm{disc}(L)$

in this case). The form f_L is said to be a *reduced binary quadratic form* if the following inequalities are satisfied:

$$0 < b < \sqrt{\mathrm{disc}(f_L)} < \min\left(b + |a|, b + |c|\right). \tag{5.1}$$

It is well-known that every indefinite binary quadratic form is properly equivalent to at least one reduced binary quadratic form, see [CS99, §15.3.3]. Using this fact, we will prove the following result for $L_{2\text{-}8}$:

Lemma 5.2. *The lattice $L_{2\text{-}8}$ is the unique indefinite even lattice of rank 2 and discriminant 8. In particular, $L_{2\text{-}8}$ is unique in its genus.*

Proof. We use the correspondence between binary quadratic forms and even lattices of rank 2. For the first statement, it suffices to show that there exists a unique reduced binary quadratic form of discriminant 8, up to equivalence. Let $f = ax^2 + bxy + cy^2$ be a reduced binary quadratic form of discriminant $\mathrm{disc}(f) = 8$. By (5.1), we have $b = 1$ or $b = 2$. However, since $b^2 - 4ac = 8$, we have $b^2 \equiv 0 \pmod 4$, thus it follows that $b = 2$. Again using $b^2 - 4ac = 8$, we find $ac = -1$, and so $(a, c) = (\pm 1, \mp 1)$. However, the binary quadratic forms $x^2 + 2xy - y^2$ and $-x^2 + 2xy + y^2$ are clearly properly equivalent, so there exists a unique equivalence class of reduced binary quadratic forms of discriminant 8, as required.

For the final assertion, suppose L is any lattice in the same genus as $L_{2\text{-}8}$. Then L is an even, indefinite lattice of rank 2 and discriminant $\mathrm{disc}(L) = 8$, so we have $L \simeq L_{2\text{-}8}$ by the above discussion. □

For $L_{2\text{-}6}$, the situation is slightly more delicate:

Lemma 5.3. *There are precisely 2 even, indefinite lattices of rank 2 and discriminant 12, and they are not in the same genus. In particular, $L_{2\text{-}6}$ is unique in its genus.*

Proof. We use the same strategy as in the proof of Lemma 5.2. If $f = ax^2 + bxy + cy^2$ is a reduced binary quadratic form of discriminant 12, then (5.1) implies that $b = 1$, $b = 2$, or $b = 3$. Once again we can reduce the equality $b^2 - 4ac = 12$ modulo 4 to rule out $b = 1$ and $b = 3$. Thus, we have $b = 2$ and $ac = -2$. This leads to two possibly inequivalent reduced binary quadratic forms, whose lattices are given by

$$L_1 := \begin{pmatrix} 2 & 2 \\ 2 & -4 \end{pmatrix}, \qquad L_2 := \begin{pmatrix} -2 & 2 \\ 2 & 4 \end{pmatrix}.$$

We claim that $L_1 \not\simeq L_2$. In fact, we will show that L_1 and L_2 are not in the same genus, because they do not have isometric discriminant groups. Firstly, note that A_{L_1} and A_{L_2} are isomorphic as groups, as they are each isomorphic to the group $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z}$. A straightforward but technical check shows that the 3-primary part $A_{L_1}^{(3)}$ of A_{L_1} is a group of order 3 for which any generator $x \in A_{L_1}^{(3)}$ has $q(x) = \frac{4}{3} \pmod{2\mathbf{Z}}$, whereas any generator $y \in A_{L_2}^{(3)}$ has $q(y) = \frac{2}{3} \pmod{2\mathbf{Z}}$. Thus, we find $A_{L_1} \not\simeq A_{L_2}$, hence L_1 and L_2 are not in the same genus, and there exist precisely two even, indefinite lattices of rank 2 and discriminant 12. □

Remark 5.4. One can check that L_1 is isometric to $L_{2\text{-}6}$. Indeed, if $v, w \in L_{2\text{-}6}$ are basis vectors whose Gram matrix is

$$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix},$$

then $v, w - v \in L_{2\text{-}6}$ are basis vectors whose Gram matrix is

$$\begin{pmatrix} 2 & 2 \\ 2 & -4 \end{pmatrix}.$$

We immediately obtain the following.

Proposition 5.5. *The lattices $L_{2\text{-}6}$ and $L_{2\text{-}8}$ are each unique in their respective genera.*

Proof. Combine Lemma 5.2 and Lemma 5.3. □

Proposition 5.6. *Let Z be a K3 surface for which $\mathrm{NS}(Z)$ is isometric to either $L_{2\text{-}6}$ or $L_{2\text{-}8}$. Then Z has no non-trivial Fourier–Mukai partners.*

Proof. Since $L_{2\text{-}6}$ and $L_{2\text{-}8}$ are unique in their genera, the Counting Formula of Theorem 2.21 consists of a single term:

$$|\operatorname{FM}(Z)| = \left|O(\mathrm{NS}(Z))\backslash O(A_{\mathrm{NS}(Z)})/O_{\mathrm{Hodge}}(T(Z))\right|. \tag{5.2}$$

With a straightforward computation, one can show that $O(A_{\mathrm{NS}(Z)}) = \{\pm\mathrm{id}\}$, thus the natural map $O(\mathrm{NS}(Z)) \to O(A_{\mathrm{NS}(Z)})$ is surjective and the double quotient on the right-hand side of (5.2) is trivial. Therefore, we have $|\operatorname{FM}(Z)| = 1$, as required. □

5.2. **Family 3-1.** Our strategy for proving Theorem 5.1 for Family 3-1 is inspired by [MS24]. We note that the branch divisor Z is an elliptic K3 surface of Picard rank 3. In that paper, only elliptic K3 surfaces of Picard rank 2 are considered, but we will show that the techniques can be extended to Z.

We begin by recalling certain facts from [MS24]. To any elliptic K3 surface $f\colon S \to \mathbf{P}^1$, we can associate a relative Jacobian which we denote by $\mathrm{J}^0(S) \to \mathbf{P}^1$. Then, by [MS24, Lemma 2.8], we have a short exact sequence

$$0 \longrightarrow T(S) \longrightarrow T(\mathrm{J}^0(S)) \xrightarrow{\alpha_S} \mathbf{Z}/t\mathbf{Z} \longrightarrow 0, \tag{5.3}$$

where t is the multisection index of f. Via Lemma 2.17, the quotient $T(\mathrm{J}^0(S))/T(S)$ can be seen as a subgroup of $A_{\mathrm{NS}(S)}$. Explicitly, this subgroup is cyclic and isotropic, generated by the class $\frac{1}{t}F \in A_{\mathrm{NS}(S)}$, where F is the fibre class of the elliptic fibration f.

Moreover, the map α_S of (5.3) is the element of $\mathrm{Br}(\mathrm{J}^0(S)) \simeq \mathrm{Hom}(T(\mathrm{J}^0(S)), \mathbf{Q}/\mathbf{Z})$ which corresponds to the class of S via the natural isomorphism $\mathrm{Br}(S) \simeq Ш(\mathrm{J}^0(S))$. Here $Ш(\mathrm{J}^0(S))$ denotes the *Tate–Šafarevič group* of $\mathrm{J}^0(S)$.

Proposition 5.7. *Let Z be a K3 surface with $\mathrm{NS}(Z) \simeq L_{3\text{-}1}$. Then Z has no non-trivial Fourier–Mukai partners.*

Proof. Suppose Z' is a Fourier–Mukai partner of Z. We wish to show that $Z' \simeq Z$.

Firstly, note that Z' admits an elliptic fibration $f\colon Z' \to \mathbf{P}^1$ by [HT17, Proposition 16]. Denote by $\mathrm{J}^0(Z')$ the relative Jacobian of f. Fix any Hodge isometry $\phi\colon T(Z) \simeq T(Z')$, which exists due to the derived Torelli theorem, see Theorem 2.19. Then the composition

$$T(Z) \overset{\phi}{\simeq} T(Z') \hookrightarrow T(\mathrm{J}^0(Z'))$$

is an overlattice of $T(Z)$, hence $H := T(\mathrm{J}^0(Z'))/T(Z)$ naturally defines an isotropic subgroup of $A_{\mathrm{NS}(Z)}$ via Lemma 2.17. Moreover, we have

$$16 = \det(T(Z)) = |H|^2 \det(T(\mathrm{J}^0(Z))),$$

thus H has order $1, 2$ or 4. However, $A_{\mathrm{NS}(Z')} \simeq A_{3\text{-}1}$ has no isotropic subgroups of order 4 by Lemma 4.19. On the other hand if H is the trivial group, then the multisection index of f is 1. In this case Z' has no non-trivial Fourier–Mukai partners by [HLOY04, Corollary

2.7]. However, by assumption, Z is a Fourier–Mukai partner of Z' and Z admits no elliptic fibrations of multisection index 1, a contradiction. We conclude that H must be an isotropic subgroup of order 2. By Lemma 4.19, $A_{\mathrm{NS}(Z)}$ has precisely 3 isotropic subgroups of order 2, namely those appearing in (4.7). We assume that $H = \langle \frac{1}{2} h_1 \rangle$, as the other two cases follow by symmetry. Then, if we denote by $\mathrm{J}^0(Z)$ the relative Jacobian of the elliptic fibration $f_1 \colon Z \to \mathbf{P}^1$ defined by h_1, we have a commutative diagram by Lemma 2.17:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(Z) & \longrightarrow & T(\mathrm{J}^0(Z)) & \xrightarrow{\alpha_Z} & \mathbf{Z}/2\mathbf{Z} & \longrightarrow & 0 \\ & & \simeq \downarrow \phi & & \simeq \downarrow \tilde{\phi} & & \simeq \downarrow & & \\ 0 & \longrightarrow & T(Z') & \longrightarrow & T(\mathrm{J}^0(Z')) & \xrightarrow{\alpha_{Z'}} & \mathbf{Z}/2\mathbf{Z} & \longrightarrow & 0. \end{array} \tag{5.4}$$

Since Z has Picard rank 3, we have $O_{\mathrm{Hodge}}(T(Z)) = \{\pm \mathrm{id}\}$ by Lemma 2.22. Hence, up to a sign, $\tilde{\phi}$ is the pullback along a group isomorphism $g \colon \mathrm{J}^0(Z') \simeq \mathrm{J}^0(Z)$. We see from (5.4) that the isomorphism g satisfies $g^*\alpha_Z = \alpha_{Z'}$, thus we have $Z \simeq Z'$ by [MS24, Proposition 4.8]. □

6. Equivariant Kuznetsov components and semiorthogonal decompositions

We now study the equivariant Kuznetsov components of the Fano varieties in the three families under consideration. The results in this section hold in greater generality than Theorem 4.1 and Theorem 5.1, which hold for Z-general Fano threefolds. Let X be a Fano threefold in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$. Recall from Definition 3.6 that we say that X is *K3-general* if the branch locus Z of the double cover $X \to Y$ is a smooth K3 surface. Note that Z-general Fano threefolds are always K3-general, but the converse does not hold, see also Remark 3.7.

The main goal of this section is to prove the following two results.

Theorem 6.1. *Let X be a K3-general Fano variety in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$, and let X' be a Fano variety in the same family. Then any equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ lifts to an equivalence of equivariant categories $\mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}$:*

$$\mathcal{A}_X \simeq \mathcal{A}_{X'} \implies \mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}.$$

Proof. This follows from Proposition 6.4, Proposition 6.8, and Proposition 6.15 below, and [DJR25, Lemma 6.2]. An analogous result was also used in the proof of [JLLZ24, Theorem 9.9]. □

Theorem 6.2. *Let X be a K3-general Fano variety in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$. Let Z be the associated K3 surface of X. Then we have an equivalence*

$$\mathcal{A}_X^{\mu_2} \simeq \mathrm{D^b}(Z).$$

Proof. This follows from Proposition 6.3, Proposition 6.7, and Proposition 6.14 below. □

6.1. Family 2-6(b).

Let X be a K3-general Fano variety in Family 2-6(b). That is, X is a double cover of a bidegree $(1,1)$-divisor $Y \subset \mathbf{P}^2 \times \mathbf{P}^2$ branched in an anticanonical K3 surface $Z \subset Y$. Recall from Lemma 4.10 that Y can equivalently be described as $\mathbf{P}(T_{\mathbf{P}^2})$, the projectivised tangent bundle of the projective plane. By Orlov's Projective Bundle Formula [Orl92], we have the semiorthogonal decomposition

$$\mathrm{D^b}(\mathbf{P}(T_{\mathbf{P}^2})) = \langle \pi^*\mathrm{D^b}(\mathbf{P}^2), \pi^*\mathrm{D^b}(\mathbf{P}^2) \otimes \mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1) \rangle \tag{6.1}$$

where $\pi \colon Y \to \mathbf{P}^2$ is the projective bundle map, and $\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1)$ is the tautological bundle on the projectivisation. We note that (6.1) is a rectangular Lefschetz decomposition with $\mathcal{B} = \pi^*\mathrm{D^b}(\mathbf{P}^2)$, see Definition 2.5.

By Lemma 2.12, we have

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \pi^*\mathrm{D^b}(\mathbf{P}^2)\rangle. \tag{6.2}$$

Moreover, [KP17, Theorem 1.1] directly applies, and gives us the following description of the equivariant Kuznetsov component.

Proposition 6.3. *We have*

$$\mathcal{A}_X^{\mu_2} \simeq \mathrm{D^b}(Z).$$

We also have an explicit description of the involution autoequivalence $\tau_{\mathcal{A}_X} : \mathcal{A}_X \to \mathcal{A}_X$ in this case:

Proposition 6.4. *We have the isomorphism of functors*

$$S_{\mathcal{A}_X} \simeq \tau_{\mathcal{A}_X}[2].$$

Proof. This is by either [Kuz19, Corollary 3.18], or [KP17, Theorem 7.7 and Proposition 7.10], or Proposition 2.14. □

6.2. **Family 2-8.** Let X be a K3-general Fano variety in Family 2-8. That is, X is the double cover of $Y := \mathrm{Bl}_p(\mathbf{P}^3)$ branched in an anticanonical K3 surface $Z \subset Y$. Recall that the image of the composition $Z \hookrightarrow Y \to \mathbf{P}^3$ is a singular quartic K3 surface $Z' \subset \mathbf{P}^3$ with an ordinary double point at p. If we let X' be the double cover of $\mathbf{P}^3$ branched in Z', then we have $X = \mathrm{Bl}_p(X')$. In other words, we are in the setting of [IK15, §3.2].

If we denote by $H \in \mathrm{NS}(Y)$ the pullback of a hyperplane class, and by $E \in \mathrm{NS}(Y)$ the exceptional divisor, then the canonical bundle of Y is $K_Y = 2E - 4H$. Let $D = 2H - E$, so that $K_Y = -2D$. Then, by [IK15, Proposition 3.6], we obtain the following semiorthogonal decomposition of $\mathrm{D^b}(Y)$:

$$\begin{aligned}\mathrm{D^b}(Y) &= \langle \mathcal{O}_Y(-3H+E), \mathcal{O}_Y(-2H), \mathcal{O}_Y(-2H+E), \mathcal{O}_Y(-H), \mathcal{O}_Y(-E), \mathcal{O}_Y\rangle \\ &= \langle \mathcal{E}_1(-D), \mathcal{E}_2(-D), \mathcal{E}_3(-D), \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\rangle,\end{aligned} \tag{6.3}$$

where

$$\mathcal{E}_1 := \mathcal{O}_Y(-H), \quad \mathcal{E}_2 := \mathcal{O}_Y(-E), \quad \mathcal{E}_3 := \mathcal{O}_Y.$$

Applying Proposition 2.13, we obtain the following semiorthogonal decomposition of $\mathrm{D^b}(X)$.

Lemma 6.5. *Let $f : X \to Y$ be as above. Then there is a semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X(-H), \mathcal{O}_X(-E), \mathcal{O}_X\rangle. \tag{6.4}$$

Now, using Lemma 2.8, we obtain the following semiorthogonal decomposition of the equivariant derived category.

Lemma 6.6. *We have the semiorthogonal decomposition*

$$\mathrm{D^b}(X)^{\mu_2} = \langle \mathcal{A}_X^{\mu_2}, \mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0, \mathcal{O}_X(-H)\rho_1, \mathcal{O}_X(-E)\rho_1, \mathcal{O}_X\rho_1\rangle.$$

Proof. Apply Lemma 2.8 to the semiorthogonal decomposition from Lemma 6.5. □

Recall from Theorem 2.11 that $\mathrm{D^b}(X)^{\mu_2}$ admits another semiorthogonal decomposition given as

$$\mathrm{D^b}(X)^{\mu_2} = \langle f_0^*\mathrm{D^b}(Y), j_{0*}\mathrm{D^b}(Z)\rangle.$$

Combining this with the semiorthogonal decomposition (6.3) of $\mathrm{D^b}(Y)$, we obtain

$$\begin{aligned}\mathrm{D^b}(X)^{\mu_2} &= \langle f_0^*\mathrm{D^b}(Y), j_{0*}\mathrm{D^b}(Z)\rangle \\ &= \langle \mathcal{O}_X(-3H+E)\rho_0, \mathcal{O}_X(-2H)\rho_0, \mathcal{O}_X(-2H+E)\rho_0, \\ &\quad \mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0, j_{0*}\mathrm{D^b}(Z)\rangle.\end{aligned} \tag{6.5}$$

We now show that the semiorthogonal decomposition of $\mathrm{D^b}(X)^{\mu_2}$ of Lemma 6.6 is a mutation of (6.5), which allows us to compute the equivariant Kuznetsov component $\mathcal{A}_X^{\mu_2}$.

Proposition 6.7. *We have*

$$\mathcal{A}_X^{\mu_2} \simeq \mathrm{D^b}(Z).$$

Proof. We apply Lemma 2.2 to the semiorthogonal decomposition (6.5), with

$$\mathcal{A}_1 = \langle \mathcal{O}_X(-3H+E)\rho_0, \mathcal{O}_X(-2H)\rho_0, \mathcal{O}_X(-2H+E)\rho_0 \rangle$$

and

$$\mathcal{A}_2 = \langle \mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0, j_{0*}\mathrm{D^b}(Z) \rangle.$$

Note that $S_{\mathrm{D^b}(X)^{\mu_2}}(-) = - \otimes \mathcal{O}_X(-2H+E)\rho_1$. Thus, we get

$$\begin{aligned} \mathrm{D^b}(X)^{\mu_2} &= \langle \mathcal{A}_2, S^{-1}_{\mathrm{D^b}(X)^{\mu_2}}(\mathcal{A}_1) \rangle \\ &= \langle \mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0, j_{0*}\mathrm{D^b}(Z), \mathcal{O}_X(-H)\rho_1, \mathcal{O}_X(-E)\rho_1, \mathcal{O}_X\rho_1 \rangle \\ &= \langle \mathbf{L}_{\mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0} j_{0*}\mathrm{D^b}(Z), \mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0, \\ &\qquad \mathcal{O}_X(-H)\rho_1, \mathcal{O}_X(-E)\rho_1, \mathcal{O}_X\rho_1 \rangle. \end{aligned}$$

This exhibits $\mathbf{L}_{\mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0} j_{0*}\mathrm{D^b}(Z) \simeq \mathrm{D^b}(Z)$ as the right-orthogonal of the subcategory

$$\langle \mathcal{O}_X(-H)\rho_0, \mathcal{O}_X(-E)\rho_0, \mathcal{O}_X\rho_0, \mathcal{O}_X(-H)\rho_1, \mathcal{O}_X(-E)\rho_1, \mathcal{O}_X\rho_1 \rangle,$$

which is precisely $\mathcal{A}_X^{\mu_2}$, by Lemma 6.6. □

Proposition 6.8. *We have the isomorphism of functors*

$$S_{\mathcal{A}_X} \simeq \tau_{\mathcal{A}_X}[2].$$

Proof. This is by [IK15, Corollary 3.7]. □

6.3. **Family 3-1.** Recall that the Fano threefolds X in Family 3-1 are double covers of $Y = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ branched in divisors Z of tridegree $(2,2,2)$. In this subsection, we assume that X is a K3-general Fano threefold in Family 3-1, so that Z is a smooth K3 surface. We first compute some mutations that we will need later on.

Lemma 6.9. *We have*

$$\begin{aligned} \mathbf{R}_{\mathcal{O}_Y(0,0,0)}\mathcal{O}_Y(-1,0,0) &\simeq \mathcal{O}_Y(1,0,0)[-1] \\ \mathbf{R}_{\mathcal{O}_Y(0,0,0)}\mathcal{O}_Y(0,-1,0) &\simeq \mathcal{O}_Y(0,1,0)[-1] \\ \mathbf{R}_{\mathcal{O}_Y(0,0,0)}\mathcal{O}_Y(0,0,-1) &\simeq \mathcal{O}_Y(0,0,1)[-1]. \end{aligned}$$

Proof. We show the first isomorphism. For the mutation in question, the relevant Hom-space is the dual of

$$\begin{aligned} \mathrm{Hom}^\bullet(\mathcal{O}_Y(-1,0,0), \mathcal{O}_Y(0,0,0)) &= \mathrm{Hom}^\bullet(\mathcal{O}_Y(0,0,0), \mathcal{O}_Y(1,0,0)) \\ &\simeq H^\bullet(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) \otimes H^\bullet(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) \otimes H^\bullet(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) \\ &= \mathbf{C}^2[0] \end{aligned}$$

where for the second isomorphism, we have used the Künneth Theorem for sheaf cohomology. Hence, the right mutation fits into the triangle

$$\mathbf{R}_{\mathcal{O}_Y(0,0,0)}\mathcal{O}_Y(-1,0,0) \to \mathcal{O}_Y(-1,0,0) \to \mathcal{O}_Y(0,0,0)^{\oplus 2}.$$

Now consider the Euler short exact sequence $0 \to \mathcal{O}_{\mathbf{P}^1}(-1) \to \mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbf{P}^1}(1) \to 0$ on the first $\mathbf{P}^1$ factor. Pulling this back to Y and comparing with the triangle above gives the desired isomorphism. The rest of the isomorphisms are similar. □

Lemma 6.10. *We have the semiorthogonal decomposition*

$$\begin{aligned}\mathrm{D^b}(Y) = \langle &\mathcal{O}_Y(-1,-1,-1), \mathcal{O}_Y(0,-1,-1), \mathcal{O}_Y(-1,0,-1), \mathcal{O}_Y(-1,-1,0),\\ &\mathcal{O}_Y(0,0,0), \mathcal{O}_Y(1,0,0), \mathcal{O}_Y(0,1,0), \mathcal{O}_Y(0,0,1)\rangle.\end{aligned}$$

Proof. We have

$$\begin{aligned}\mathrm{D^b}(Y) &= \langle \mathrm{D^b}(\mathbf{P}^1) \boxtimes \mathrm{D^b}(\mathbf{P}^1) \boxtimes \mathrm{D^b}(\mathbf{P}^1)\rangle\\ &= \langle \mathcal{O}_Y(i,j,k)\rangle_{i,j,k\in\{-1,0\}}\\ &= \langle \mathcal{O}_Y(-1,-1,-1), \mathcal{O}_Y(0,-1,-1), \mathcal{O}_Y(-1,0,-1), \mathcal{O}_Y(-1,-1,0),\\ &\qquad \mathcal{O}_Y(-1,0,0), \mathcal{O}_Y(0,-1,0), \mathcal{O}_Y(0,0,-1), \mathcal{O}_Y(0,0,0)\rangle\end{aligned}$$

by [Kuz11, Theorem 5.8]. Now apply Lemma 6.9 to get the desired semiorthogonal decomposition. ☐

Set

$$\mathcal{E}_1 := \mathcal{O}_Y(0,0,0), \quad \mathcal{E}_2 := \mathcal{O}_Y(1,0,0), \quad \mathcal{E}_3 := \mathcal{O}_Y(0,1,0), \quad \mathcal{E}_4 := \mathcal{O}_Y(0,0,1)$$

with $\underline{\mathcal{E}} = \{\mathcal{E}_1, \dots, \mathcal{E}_4\}$. Also set $\mathbf{1} = \mathcal{O}_Y(1,1,1)$. Then clearly $\underline{\mathcal{E}}$ is an exceptional collection in $\mathrm{D^b}(Y)$, as is $\{\underline{\mathcal{E}}(-\mathbf{1}), \underline{\mathcal{E}}\} = \mathrm{D^b}(Y)$ by Lemma 6.10. Then by Proposition 2.13, we get following semiorthogonal decomposition.

Lemma 6.11. *We have the semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X(0,0,0), \mathcal{O}_X(1,0,0), \mathcal{O}_X(0,1,0), \mathcal{O}_X(0,0,1)\rangle.$$

From Lemma 6.11, we obtain the first semiorthogonal decomposition of $\mathrm{D^b}(X)^{\mu_2}$.

Lemma 6.12. *We have the semiorthogonal decomposition*

$$\begin{aligned}\mathrm{D^b}(X)^{\mu_2} = \langle &\mathcal{A}_X^{\mu_2}, \mathcal{O}_X(0,0,0)\rho_0, \mathcal{O}_X(1,0,0)\rho_0, \mathcal{O}_X(0,1,0)\rho_0, \mathcal{O}_X(0,0,1)\rho_0\\ &\mathcal{O}_X(0,0,0)\rho_1, \mathcal{O}_X(1,0,0)\rho_1, \mathcal{O}_X(0,1,0)\rho_1, \mathcal{O}_X(0,0,1)\rho_1\rangle.\end{aligned}$$

Proof. Apply Theorem 2.8 to the semiorthogonal decomposition from Lemma 6.11. ☐

The second semiorthogonal decomposition of $\mathrm{D^b}(X)^{\mu_2}$ comes from Theorem 2.11.

Lemma 6.13. *We have the semiorthogonal decomposition*

$$\begin{aligned}\mathrm{D^b}(X)^{\mu_2} &= \langle f_0^*\mathrm{D^b}(Y), j_{0*}\mathrm{D^b}(Z)\rangle\\ &= \langle \mathcal{O}_X(-1,-1,-1)\rho_0, \mathcal{O}_X(0,-1,-1)\rho_0, \mathcal{O}_X(-1,0,-1)\rho_0, \mathcal{O}_X(-1,-1,0)\rho_0\\ &\qquad \mathcal{O}_X(-1,0,0)\rho_0, \mathcal{O}_X(0,-1,0)\rho_0, \mathcal{O}_X(0,0,-1)\rho_0, \mathcal{O}_X(0,0,0)\rho_0, j_{0*}\mathrm{D^b}(Z)\rangle.\end{aligned}$$

Proof. This is an instance of Theorem 2.11 combined with Lemma 6.10. ☐

Proposition 6.14. *We have*

$$\mathcal{A}_X^{\mu_2} \simeq \mathrm{D^b}(Z).$$

Proof. We apply Lemma 2.2 to the semiorthogonal decomposition from Lemma 6.13 with

$$\mathcal{A}_1 = \langle \mathcal{O}_X(-1,-1,-1)\rho_0, \mathcal{O}_X(0,-1,-1)\rho_0, \mathcal{O}_X(-1,0,-1)\rho_0, \mathcal{O}_X(-1,-1,0)\rho_0\rangle$$

and

$$\mathcal{A}_2 = \langle \mathcal{O}_X(-1,0,0)\rho_0, \mathcal{O}_X(0,-1,0)\rho_0, \mathcal{O}_X(0,0,-1)\rho_0, \mathcal{O}_X(0,0,0)\rho_0, j_{0*}\mathrm{D^b}(Z)\rangle.$$

We also write

$$\mathcal{A}_2' := {}^\perp(j_{0*}\mathrm{D^b}(Z)) \subset \mathcal{A}_2.$$

Note that $S_{\mathrm{D^b}(X)^{\mu_2}}(-) = - \otimes \mathcal{O}_X(-1,-1,-1)\rho_1$. We now mutate $\mathcal{A}_2$ to the left along $\mathcal{A}_1$, and then mutate $j_{0*}\mathrm{D^b}(Z)$ to the left along $\mathcal{A}_2'$:

$$\begin{aligned}\mathrm{D^b}(X)^{\mu_2} &= \langle \mathcal{A}_2, S^{-1}_{\mathrm{D^b}(X)^{\mu_2}}(\mathcal{A}_1)\rangle \\ &= \langle \mathcal{O}_X(-1,0,0)\rho_0, \mathcal{O}_X(0,-1,0)\rho_0, \mathcal{O}_X(0,0,-1)\rho_0, \mathcal{O}_X(0,0,0)\rho_0, j_{0*}\mathrm{D^b}(Z), \\ &\quad \mathcal{O}(0,0,0)\rho_1, \mathcal{O}(1,0,0)\rho_1, \mathcal{O}(0,1,0)\rho_1, \mathcal{O}(0,0,1)\rho_1\rangle \\ &= \langle \mathbf{L}_{\mathcal{A}_2'} j_{0*}\mathrm{D^b}(Z), \mathcal{O}_X(-1,0,0)\rho_0, \mathcal{O}_X(0,-1,0)\rho_0, \mathcal{O}_X(0,0,-1)\rho_0, \mathcal{O}_X(0,0,0)\rho_0 \\ &\quad \mathcal{O}_X(0,0,0)\rho_1, \mathcal{O}_X(1,0,0)\rho_1, \mathcal{O}_X(0,1,0)\rho_1, \mathcal{O}_X(0,0,1)\rho_1\rangle \\ &= \langle \mathbf{L}_{\mathcal{A}_2'} j_{0*}\mathrm{D^b}(Z), \mathcal{O}_X(0,0,0)\rho_0, \mathcal{O}_X(1,0,0)\rho_0, \mathcal{O}_X(0,1,0)\rho_0, \mathcal{O}_X(0,0,1)\rho_0 \\ &\quad \mathcal{O}_X(0,0,0)\rho_1, \mathcal{O}_X(1,0,0)\rho_1, \mathcal{O}_X(0,1,0)\rho_1, \mathcal{O}_X(0,0,1)\rho_1\rangle\end{aligned}$$

For the last equality, we have applied Lemma 6.9 three times. The right-hand-sides of the above and the semiorthogonal decomposition from Lemma 6.12 are the same, and $\mathbf{L}_{\mathcal{A}_2'}$ is a fully faithful embedding, hence we conclude the result. □

Finally, we show the relation between the Serre functor and the involution on $\mathcal{A}_X$.

Proposition 6.15. *We have*

$$S_{\mathcal{A}_X} \simeq \tau_{\mathcal{A}_X}[2].$$

Proof. Since $\mathrm{D^b}(Y) = \langle \underline{\mathcal{E}}(-\mathbf{1}), \underline{\mathcal{E}}\rangle$ by Lemma 6.10, the collection $\{\underline{\mathcal{E}}(-\mathbf{1}), \underline{\mathcal{E}}\}$ is full in $\mathrm{D^b}(Y)$. Thus, by Proposition 2.14, we get the desired isomorphism of functors. □

7. Categorical Torelli Theorems

We are now ready to put all of the previous results together to prove Theorem A. Recall that a Fano threefold X in one of the three families under consideration is called Z-general if the Néron–Severi lattice of the branch divisor Z is isometric to $L_{2\text{-}6}$, $L_{2\text{-}8}$, or $L_{3\text{-}1}$, see Definition 3.6.

Theorem 7.1. *Let X be a Z-general Fano variety in one of the families $\mathcal{X}_{2\text{-}6(b)}$, $\mathcal{X}_{2\text{-}8}$, and $\mathcal{X}_{3\text{-}1}$, and let X' be a Fano variety in the same family. Then, if there exists an equivalence between the Kuznetsov components $\mathcal{A}_X$ and $\mathcal{A}_{X'}$, it follows that X is isomorphic to X':*

$$\mathcal{A}_X \simeq \mathcal{A}_{X'} \implies X \simeq X'.$$

Proof. Let Z and Z' be the associated K3 surfaces of X and X'. By Theorem 6.1, the equivalence Φ lifts to an equivalence of equivariant categories $\Phi^{\mu_2}\colon \mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}$. By Theorem 6.2, this induces an equivalence $\mathrm{D^b}(Z) \simeq \mathrm{D^b}(Z')$. This equivalence is Fourier–Mukai by [Orl97]. In particular, Z and Z' are Fourier–Mukai partners. By Theorem 5.1, this means that we must have $Z \simeq Z'$. Finally, we conclude that $X \simeq X'$ by Theorem 4.1. □

7.1. Open problems.

7.1.1. *Verra fourfolds.* Fano threefolds in Family 2-6(a) and Family 2-6(b) are usually called *ordinary* and *special Verra threefolds*, respectively. They were first studied in [Ver04]. A Fano variety X in Family 2-6(a) (so an ordinary Verra threefold) is a bidegree $(2,2)$ divisor in $\mathbf{P}^2 \times \mathbf{P}^2$. To such a Fano variety one associates two sextic K3 surfaces, just as in Family 2-6(b). Explicitly, let V be the double cover of $\mathbf{P}^2 \times \mathbf{P}^2$ branched in X. The fourfold V is a so-called *Verra fourfold*. The discriminant loci of the two projections $V \to \mathbf{P}^2$ are sextic curves in $\mathbf{P}^2$, and the associated double covers are sextic structures on two K3 surfaces S_1, S_2. We showed in Section 4 that a Verra threefold in Family 2-6(b) also induces two sextic structures, which in this case are defined on the same K3 surface. However, the K3 surfaces S_1 and S_2 are well-known for being L-equivalent under certain conditions, cf. [KS18, §2.6.2], but not isomorphic [KKM20]. In light of the conjectural relationship between L- and D-equivalence for K3 surfaces [KS18, Mei26], the categorical Torelli question for Verra fourfolds seems like a particularly interesting open case.

7.1.2. *Family 2-8(b).* Recall from Remark 3.8 that Family 2-8 splits into two subfamilies, called 2-8(a) and 2-8(b). Theorem 7.1 holds for Z-general Fano threefolds in Family 2-8. These are all members of Family 2-8(a). A Fano threefold in Family 2-8 is a member of Family 2-8(b) if the scheme-theoretic intersection $Z \cap E$ is singular but reduced. In the smooth case, the exceptional divisor $e \in \mathrm{NS}(Z)$ always satisfies $e^2 = 2$. However, if $Z \cap E$ is not smooth, the Néron–Severi lattice of Z depends on the singularities of $Z \cap E$. Thus, to prove a categorical Torelli theorem for very general members of Family 2-8(b), more Néron–Severi lattices need to be considered.

7.1.3. *The remaining double covers.* Finally, as we already noted in the introduction, there are only two families of Fano threefolds of Picard rank $\rho \geq 2$ which are double covers left for which a categorical Torelli theorem has not been proved, namely Families 2-2 and 2-18 on Fanography [Bel26].

(1) **Family 2-2:** These are double covers of $\mathbf{P}^1 \times \mathbf{P}^2$ branched in a divisor of bidegree $(2, 4)$. The approach of the current paper is not possible, since the canonical bundle of Z is no longer trivial. In fact, $K_Z \simeq \mathcal{O}_Z(0, 1)$, so Z is of general type. By the trichotomy discussed at the end of [DJR25, Section 3], one might expect the positivity of K_Z to mean that the equivariant Kuznetsov component should contain a copy of $\mathrm{D^b}(Z)$. One could then apply the techniques of [DJR25] to show that an equivalence of Kuznetsov components implies a Hodge isometry between the middle cohomologies of the respective branch divisors. However, we are unaware of a Torelli theorem for bidegree $(2, 4)$ divisors in $\mathbf{P}^1 \times \mathbf{P}^2$.
(2) **Family 2-18:** These are double covers of $\mathbf{P}^1 \times \mathbf{P}^2$ branched in a divisor of bidegree $(2, 2)$. In this case, $K_Z \simeq \mathcal{O}_Z(0, -1)$, i.e. Z is rational, but not Fano. By the trichotomy from [DJR25, Section 3], the negativity of K_Z might lead one to expect $\mathrm{D^b}(Z)$ to contain the equivariant Kuznetsov component.

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