

CYCLIC COVERS: HODGE THEORY AND CATEGORICAL TORELLI THEOREMS

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ABSTRACT. Let Y admit a rectangular Lefschetz decomposition of its derived category, and consider a cyclic cover $X \rightarrow Y$ ramified over a divisor Z . In a setting not considered by Kuznetsov and Perry [KP17], we define a subcategory \mathcal{A}_Z of the equivariant derived category of X which contains, rather than is contained in, $D^b(Z)$. We then show that the equivariant category of the Kuznetsov component of X is decomposed into copies of \mathcal{A}_Z .

As an application, we relate \mathcal{A}_Z with the cohomology of Z under some numerical assumptions. In particular, we obtain categorical Torelli theorems for the lowest degree prime Fano threefolds of index 1 and 2.

1. INTRODUCTION

In [Kuz09], Kuznetsov systematically studies derived categories of Fano threefolds and subcategories – known as *Kuznetsov components* – arising as orthogonal complements to exceptional collections. These categories are linked to questions of rationality [Kuz10], stability conditions [BLMS23] and to hyperkähler geometry through moduli spaces [LPZ23], as well as questions in birational geometry and Hodge theory [Per22].

In this paper, we start from a variety Y admitting a sheaf $\mathcal{O}_Y(1)$ and a rectangular Lefschetz decomposition of $D^b(Y)$, i.e. an admissible subcategory \mathcal{B} and a decomposition

$$D^b(Y) = \langle \mathcal{B}, \dots, \mathcal{B}(m-1) \rangle,$$

with $\mathcal{B}(i)$ the image of \mathcal{B} under the equivalence $-\otimes \mathcal{O}_Y(1)$. Examples of varieties with rectangular decompositions are projective space, Grassmannians, some homogeneous spaces (see [Kuz19, Section 4.1]). Another important case, also covered in this article, is when Y is a Deligne–Mumford stack, for example a weighted projective space. Then, we consider an n -fold cover X of Y , ramified along a divisor $Z \in |\mathcal{O}_Y(nd)|$. The derived category $D^b(X)$ has a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \dots, \mathcal{B}_X(M-1) \rangle$$

where \mathcal{B}_X is the pullback of \mathcal{B} under the covering map, and $M := m - (n-1)d$. The action of μ_n on X by the covering involution restricts to one on \mathcal{A}_X , giving rise to the Kuznetsov component $\mathcal{A}_X^{\mu_n} \subset D^b(X)^{\mu_n}$. This is the same setting of [KP17], with the crucial difference that we work in the range $0 < M < d$, rather than $d \leq M$. The former corresponds to allowing more positivity for Z (see Section 3.3). In [KP17], $D^b(Z)$ also admits a Kuznetsov component \mathcal{A}_Z , and $\mathcal{A}_X^{\mu_n}$ decomposes into copies of \mathcal{A}_Z . We define a subcategory $\mathcal{A}_Z \subset D^b(X)^{\mu_n}$ which *contains*, rather than is contained in, $D^b(Z)$ (see

2020 *Mathematics Subject Classification*. Primary 14F08; secondary 14J45, 14C34.

Key words and phrases. Derived categories, cyclic covers, Fano threefolds, K-theory, categorical Torelli theorems.

HD and AJ were supported by ERC Consolidator Grant WallCrossAG, no. 819864. AJ was also supported by the Luxembourg National Research Fund (FNR–17113194). FR is supported by EPSRC grant EP/R034826/1 and acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2047/1 – 390685813.

(3.2)), and in our first main result we exhibit a semiorthogonal decomposition for $\mathcal{A}_X^{\mu_n}$ with components \mathcal{A}_Z .

Theorem 1.1 (= Theorem 3.4). *Assume that we are in the situation above, and that $0 < M < d$. Then there are fully faithful functors $\Phi_j: \mathcal{A}_Z \rightarrow \mathcal{A}_X^{\mu_n}$, with $j = 0, \dots, n-2$, and a semiorthogonal decomposition:*

$$\mathcal{A}_X^{\mu_n} = \langle \Phi_0(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z) \rangle \subset \mathrm{D}^b(X)^{\mu_n}.$$

In the next part of the paper, we study the topological K-theory of \mathcal{A}_Z , and its associated Hodge structure. For this, we use Blanc's topological K-theory for categories [Bla16] and the non-commutative Hodge theory methods of [Per22] and [HLP20], slightly extending their results to admissible subcategories of global quotient DM stacks (Proposition 4.2).

We apply this to the case of very general three-dimensional double covers of weighted projective space. Here, we can make a direct comparison between $\mathrm{K}_0^{\mathrm{top}}(\mathcal{A}_Z)$ and $H^*(Z, \mathbf{Z})$, and show that primitive cohomology of Z coincides with the part $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp$ of the topological K-theory of $\mathcal{A}_X^{\mu_2}$ which is orthogonal to the algebraic classes.

Proposition 1.2 (= Proposition 5.5). *Suppose that X is a very general prime double cover of weighted projective three-dimensional space, with $0 < M \leq d$. Then $\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp$ is equipped with a Hodge structure such that the Chern character map induces a Hodge isometry*

$$\mathrm{K}_0(\mathcal{A}_X^{\mu_2})^\perp \simeq H_{\mathrm{prim}}^2(Z, \mathbf{Z}).$$

A landmark theorem by Bondal–Orlov [BO01] states that the derived category of a Fano variety X determines X up to isomorphism. This justifies the question of whether \mathcal{A}_X suffices to determine X . When this is the case we say that a *categorical Torelli theorem* holds for X . Motivated by these questions, we apply Proposition 1.2 to reconstruct primitive cohomology from categorical equivalences:

Theorem 1.3 (= Theorem 5.6). *Let X and X' be prime double covers of a weighted projective three-dimensional space, with X very general, $d = d'$ and $0 < M \leq d$. A Fourier–Mukai type equivalence $\Phi^{\mu_2}: \mathcal{A}_X^{\mu_2} \rightarrow \mathcal{A}_{X'}^{\mu_2}$ induces a Hodge isometry*

$$H_{\mathrm{prim}}^2(Z, \mathbf{Z}) \simeq H_{\mathrm{prim}}^2(Z', \mathbf{Z}).$$

Combining Theorem 1.3 with Hodge-theoretic Torelli theorems [Don83, Sai86], we prove new categorical Torelli theorems for two families of prime Fano threefolds, namely for threefolds X_2 of index 1 and genus 2, and for threefolds Y_1 of index 2 and degree 1. In the case of Y_1 , known as the *Veronese double cone*, we impose a natural condition on the equivalence (we comment on this in Section 1.1). This was the only remaining open case for index 2.

Theorem 1.4 (= Theorem 6.4). *Let X, X' be index 1, genus 2 prime Fano threefolds, with X very general. Suppose there is a Fourier–Mukai type equivalence of Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. Then $X \simeq X'$.*

Theorem 1.5 (= Theorem 6.5). *Let X, X' be Veronese double cones with X very general. Suppose that we have an equivalence $\Phi: \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$ of Fourier–Mukai type which commutes with the covering involution. Then $X \simeq X'$.*

Our methods also allow us to reprove categorical Torelli for quartic double solids, i.e. prime Fano threefolds of index 2 and degree 2 (see Theorem 6.7).

1.1. Related works. If we compare Theorem 1.1 and [KP17, Theorem 1.1] in the case where the base Y is weighted projective space, we observe that the cases $0 < M < d$, $M = d$, and $M > d$ correspond to the branch divisor Z being canonically polarized, K -trivial, or Fano (see Section 3.3). This is parallel to the trichotomy evidenced by Orlov [Orl09, Theorem 3.11] and suggests a relation between \mathcal{A}_Z and categories of matrix factorizations in the hypersurface setting.

In the case of hypersurfaces, Hirano and Ouchi proved a result analogous to Theorem 1.1 in the language of matrix factorizations [HO23, Theorem 5.6]. From a similar perspective, the remarkable paper [BFK14] investigates Hodge theoretic applications of the language of matrix factorizations. The same philosophy was recently applied to prove categorical Torelli theorems for hypersurfaces in \mathbf{P}^n [LZ23, Theorem 1.3]. A modification of this also applies to X_2 , regarded as a sextic hypersurface in $\mathbf{P}(1, 1, 1, 1, 3)$ [LZ23, Theorem 6.1]. The upcoming work [LPS23] treats this case with independent methods.

Categorical Torelli theorems have been studied for many varieties. We direct the reader to [PS23] for a survey. One possible approach to proving such theorems is to recover the Fano threefold from Bridgeland moduli spaces, as in [BMMS12, APR22, PY22, BBF⁺22, FLZ23] (for prime Fano threefolds of index 2 and degree ≥ 2) and [JLLZ21, JLZ22] (for prime Fano threefolds of index 1 and genus ≥ 6). These methods use the fact that there is a unique $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$ -orbit of Serre invariant stability conditions to show that any equivalence preserves moduli spaces of stable objects. However, for the Veronese double cone and lower genus index 1 Fano threefolds, it is not known (or even false, e.g. genus 4 [KP21, Corollary 1.9]) that such a unique orbit exists. Methods relating Hochschild cohomology and Hodge theory are developed in [HR19] and [Pir22] to prove categorical Torelli theorems for broad classes of hypersurfaces.

In [JLLZ21], a categorical Torelli theorem for special Gushel–Mukai threefolds X is shown by using the fact that the equivariant Kuznetsov component of X is equivalent (by [KP17]) to the derived category of the K3 surface that X is ramified in. Because derived equivalent K3 surfaces of this type have only one trivial Fourier–Mukai partner, the ramification divisors are isomorphic, hence the special Gushel–Mukai threefolds are isomorphic.

The additional assumption in Theorem 1.5 is equivalent to requiring that the equivalence Φ commutes with the *rotation functor* $R(-) = \mathbf{L}_{\mathcal{O}}(- \otimes \mathcal{O}_X(1))$, also called the *degree-shift functor* in [HR19] (see Remark 6.6). This assumption is natural: for example, it is necessary for a categorical Torelli theorem for cubic fourfolds [HR19, Corollary 1.2]. In fact, some special cubic fourfolds have non-trivial *Fourier–Mukai partners*, i.e. non-isomorphic cubic fourfolds sharing equivalent Kuznetsov components. In [Huy17, Section 3.2], Huybrechts showed that the number of (isomorphism classes of) Fourier–Mukai partners is finite, and that a very general cubic fourfold has no non-trivial Fourier–Mukai partners. The articles [Per21, FL23] study counts of Fourier–Mukai partners of a cubic fourfold. Inspired by these results, and by a question of Huybrechts [MS19, Question 3.25], we formulate the following question.

Question 1.6.

- (1) Do Veronese double cones have non-trivial Fourier–Mukai partners?
- (2) Do very general Veronese double cones have non-trivial Fourier–Mukai partners?
- (3) If (1) has a positive answer, then how many Fourier–Mukai partners are there up to isomorphism, and are they birational to one another?

Structure of the paper. After recalling some preliminaries and fixing notations in Section 2, we dedicate Section 3 to the study of semiorthogonal decompositions for cyclic covers

and the proof of Theorem 1.1. Section 4 is dedicated to topological K-theory and non-commutative Hodge theory. We apply these tools to the threefold setting in Section 5. The last section (Section 6) contains the categorical Torelli theorems.

Notation and conventions. We work over the field of complex numbers \mathbf{C} . Whenever we work with an arbitrary triangulated category \mathcal{C} , it will be assumed to be proper and \mathbf{C} -linear. For a projective variety (or a proper DM stack) X , we write $D(X)$ (resp. $D^b(X)$, $D_{\text{perf}}(X)$) for the derived category of (bounded, perfect) complexes of coherent sheaves on X .

Acknowledgements. We thank Arend Bayer, Pieter Belmans, Wahei Hara, Alex Perry, and Sebastian Schlegel Mejia for helpful discussions. We thank Laura Pertusi, Paolo Stellari, and Shizhuo Zhang for informing us about their work on similar topics.

2. PRELIMINARIES ON DERIVED CATEGORIES

For background on triangulated categories and derived categories, we recommend [Huy06]. Throughout this section, \mathcal{C} is a triangulated category. For objects $E, F \in \mathcal{C}$, define

$$\text{Hom}_{\mathcal{C}}^{\bullet}(E, F) := \bigoplus_{t \in \mathbf{Z}} \text{Ext}_{\mathcal{C}}^t(E, F)[-t].$$

We will omit the Hom subscripts when the category we are working in is clear from context.

Definition 2.1. An object $E \in \mathcal{C}$ is called *exceptional* if $\text{Hom}^{\bullet}(E, E) = \mathbf{C}$.

Definition 2.2. We say $\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a *semiorthogonal decomposition* of \mathcal{C} if

- (1) $\text{Hom}^{\bullet}(F, G) = 0$ for all $F \in \mathcal{A}_i, G \in \mathcal{A}_j$ if $i > j$;
- (2) for any $F \in \mathcal{C}$, there exists a sequence of morphisms

$$0 = F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 = F$$

such that $\text{Cone}(F_i \rightarrow F_{i-1}) \in \mathcal{A}_i$.

Let $i: \mathcal{A} \hookrightarrow \mathcal{C}$ be a full triangulated subcategory. If the inclusion i has a left adjoint i^* , then \mathcal{A} is called *left admissible*. If i has a right adjoint $i^!$, then \mathcal{A} is called *right admissible*. If both adjoints exist, then \mathcal{A} is called *admissible*.

Define the *right orthogonal* \mathcal{A}^{\perp} to \mathcal{A} to be the subcategory

$$\mathcal{A}^{\perp} := \{F \in \mathcal{C} \mid \text{Hom}^{\bullet}(G, F) = 0 \text{ for all } G \in \mathcal{A}\}.$$

Similarly, define the *left orthogonal* ${}^{\perp}\mathcal{A}$ to \mathcal{A} to be the subcategory

$${}^{\perp}\mathcal{A} := \{F \in \mathcal{C} \mid \text{Hom}^{\bullet}(F, G) = 0 \text{ for all } G \in \mathcal{A}\}.$$

Let $i: \mathcal{A} \hookrightarrow \mathcal{C}$ be an admissible subcategory. Let i^* and $i^!$ be the left and right adjoint, respectively. For any $F \in \mathcal{C}$, we define the *left mutation functor* $\mathbf{L}_{\mathcal{A}}(F)$ by the triangle

$$ii^!(F) \rightarrow F \rightarrow \mathbf{L}_{\mathcal{A}}(F). \quad (2.1)$$

Similarly, we define the *right mutation functor* $\mathbf{R}_{\mathcal{A}}(F)$ by the triangle

$$\mathbf{R}_{\mathcal{A}}(F) \rightarrow F \rightarrow ii^*(F). \quad (2.2)$$

In particular, when \mathcal{A} is an exceptional object E , these triangles become

$$\text{Hom}^{\bullet}(E, F) \otimes E \rightarrow F \rightarrow \mathbf{L}_E(F), \quad (2.3)$$

and

$$\mathbf{R}_E(F) \rightarrow F \rightarrow \text{Hom}^{\bullet}(F, E)^{\vee} \otimes E, \quad (2.4)$$

respectively.

Proposition 2.3. Let $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ be a semiorthogonal decomposition and let $S_{\mathcal{C}}$ be the Serre functor of \mathcal{C} . Then

$$\mathcal{C} \simeq \langle S_{\mathcal{C}}(\mathcal{C}_2), \mathcal{C}_1 \rangle \simeq \langle \mathcal{C}_2, S_{\mathcal{C}}^{-1}(\mathcal{C}_1) \rangle$$

are also semiorthogonal decompositions.

Proposition 2.4. Let $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$. Then

$$S_{\mathcal{C}_1}^{-1} = \mathbf{L}_{\mathcal{C}_2} \circ S_{\mathcal{C}}^{-1} \quad \text{and} \quad S_{\mathcal{C}_2} = \mathbf{R}_{\mathcal{C}_1} \circ S_{\mathcal{C}}.$$

2.1. Equivariant triangulated categories. The definitions of a group action on a category and the corresponding equivariant category are due to Deligne [Del97]. The following background can be found in [KP17, Section 3], which we follow. Let G be a finite group and \mathcal{C} a triangulated category.

Definition 2.5. A (right) action of G on \mathcal{C} is given by the following data:

- (1) for each $g \in G$, an autoequivalence $g^* : \mathcal{C} \rightarrow \mathcal{C}$;
- (2) for each pair $g, h \in G$, a natural isomorphism $c_{g,h} : (gh)^* \xrightarrow{\sim} h^* \circ g^*$ such that

$$\begin{array}{ccc} (fgh)^* & \xrightarrow{c_{fg,h}} & h^* \circ (fg)^* \\ c_{f,gh} \downarrow & & \downarrow h^* c_{f,g} \\ (gh)^* \circ f^* & \xrightarrow{c_{g,h} f^*} & h^* \circ g^* \circ f^* \end{array}$$

commutes for all $f, g, h \in G$.

Definition 2.6. A G -equivariant object of \mathcal{C} is a pair (F, ϕ) consisting of an object $F \in \mathcal{C}$ and a collection of isomorphisms $\phi_g : F \xrightarrow{\sim} g^*(F)$ for all $g \in G$ such that the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\phi_h} & h^*(F) & \xrightarrow{h^*(\phi_g)} & h^*(g^*(F)) \\ & \searrow \phi_{gh} & & & \uparrow c_{g,h}(F) \\ & & & & (gh)^*(F) \end{array}$$

commutes for all $g, h \in G$. The isomorphisms $\phi = \{\phi_g\}_{g \in G}$ are called the G -linearisation. The G -equivariant category \mathcal{C}^G of \mathcal{C} is the category whose objects are the G -equivariant objects of \mathcal{C} , and morphisms are those between G -invariant objects of \mathcal{C} that commute with the G -linearisations.

Example 2.7 ([Ela14, p. 12]). Suppose G is a finite abelian group. Let $\widehat{G} = \text{Hom}(G, \mathbf{C}^\times)$ be the group of irreducible representations of G . Then there is an action of \widehat{G} on \mathcal{C}^G given as follows. For every $\rho \in \widehat{G}$, we have an autoequivalence

$$\rho^*(F, (\phi_h)) := (F, (\phi_h)) \otimes \rho := (F, (\phi_h \cdot \rho(h))).$$

For $\rho_1, \rho_2 \in \widehat{G}$, the equivariant objects $\rho_1^* \rho_2^*(F, (\phi_h))$ and $(\rho_2 \circ \rho_1)^*(F, (\phi_h))$ are the same, hence we set the isomorphisms c_{ρ_2, ρ_1} to be the identities.

Theorem 2.8 ([Ela12, Theorem 6.3], [Ela14, Proposition 3.11], [KP17, Theorem 3.2]). Let X be a quasi-projective variety with an action of a finite group G . Suppose we have a semiorthogonal decomposition $D^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ where $g^* \mathcal{A}_i \subset \mathcal{A}_i$ for all $g \in G$. Then the G -equivariant category has the following semiorthogonal decomposition

$$D^b(X)^G = \langle \mathcal{A}_1^G, \dots, \mathcal{A}_n^G \rangle.$$

Definition 2.9. An action of G on \mathcal{C} is called *trivial* if for each $g \in G$ there is an isomorphism of functors, $\tau_g : \text{id} \xrightarrow{\sim} g^*$, such that $c_{g,h} \circ \tau_{gh} = h^* \tau_g \circ \tau_h$ for all $g, h \in G$.

Proposition 2.10 ([KP17, Proposition 3.3]). *Let G be a finite group acting trivially on a triangulated category \mathcal{C} . Then \mathcal{C}^G is also triangulated. Let ρ_0, \dots, ρ_n be the irreducible representations of G . Then there is a completely orthogonal¹ decomposition*

$$\mathcal{C}^G = \langle \mathcal{C} \otimes \rho_0, \dots, \mathcal{C} \otimes \rho_n \rangle.$$

In the situation of Proposition 2.10, we define the functors

$$\iota_k: \mathcal{C} \rightarrow \mathcal{C}^G, \quad F \mapsto F \otimes \rho_k; \quad (2.5)$$

$$\pi_k: \mathcal{C}^G \rightarrow \mathcal{C}, \quad F \mapsto F \otimes_{\mathcal{C}[G]} \rho_k^\vee. \quad (2.6)$$

The functors π_k are both left and right adjoint to ι_k .

3. CYCLIC COVERS AND SEMIORTHOGONAL DECOMPOSITIONS

3.1. Cyclic covers and rectangular Lefschetz decompositions. In this section we follow the notation of [KP17]. Let Y be an algebraic variety (or a proper Deligne–Mumford (DM) stack). Consider $f: X \rightarrow Y$, the degree n cyclic cover of Y ramified over a divisor Z . Denote by $j: Z \hookrightarrow X$ the embedding of Z as the ramification divisor.

Let μ_n denote the group of n -th roots of unity. This has dual group $\widehat{\mu_n} := \text{Hom}(\mu_n, \mathbf{C}^\times) \simeq \mathbf{Z}/n$. This isomorphism is given by the primitive character $\chi: \mu_n \rightarrow \mathbf{C}^\times$. Each irreducible representation ρ_i of μ_n corresponds to χ^i .

We consider the trivial action of μ_n on Y . Then f is μ_n -equivariant, and induces the following functors between equivariant derived categories:

$$\begin{aligned} f_k^*: D^b(Y) &\xrightarrow{\iota_k} D^b(Y)^{\mu_n} \xrightarrow{f^*} D^b(X)^{\mu_n}, \\ f_{k*}: D^b(X)^{\mu_n} &\xrightarrow{f_*} D^b(Y)^{\mu_n} \xrightarrow{\pi_k} D^b(Y), \\ f_k^!: D^b(Y) &\xrightarrow{\iota_k} D^b(Y)^{\mu_n} \xrightarrow{f^!} D^b(X)^{\mu_n}. \end{aligned}$$

Note that f_k^* is left adjoint to f_{k*} , and $f_k^!$ is right adjoint to f_{k*} . Similarly, define the functors $j_k^*, j_{k*}, j_k^!$ as the compositions

$$\begin{aligned} j_k^*: D^b(X)^{\mu_n} &\xrightarrow{j^*} D^b(Z)^{\mu_n} \xrightarrow{\pi_k} D^b(Z), \\ j_{k*}: D^b(Z) &\xrightarrow{\iota_k} D^b(Z)^{\mu_n} \xrightarrow{j_*} D^b(X)^{\mu_n}, \\ j_k^!: D^b(X)^{\mu_n} &\xrightarrow{j^!} D^b(Z)^{\mu_n} \xrightarrow{\pi_k} D^b(Z). \end{aligned}$$

Also note that j_k^* is left adjoint to j_{k*} , and $j_k^!$ is right adjoint to j_{k*} .

Theorem 3.1 ([KP17, Theorem 4.1]). *For each $k \in \mathbf{Z}/n$, the functors f_k^* and j_{k*} are fully faithful. Moreover, there is a semiorthogonal decomposition*

$$D^b(X)^{\mu_n} = \langle f_0^* D^b(Y), j_{0*} D^b(Z), \dots, j_{n-2*} D^b(Z) \rangle.$$

Now additionally assume that Y has a rectangular Lefschetz decomposition, i.e. assume that there is a line bundle $\mathcal{O}_Y(1)$ and an admissible subcategory $\mathcal{B} \subset D^b(Y)$ such that

$$D^b(Y) = \langle \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(m-1) \rangle$$

is a semiorthogonal decomposition, where $\mathcal{B}(t) := \mathcal{B} \otimes \mathcal{O}_Y(t)$. Assume moreover that $Z \in |\mathcal{O}_Y(nd)|$, where n, d are positive integers satisfying $0 < m - (n-1)d =: M$. Then

¹Meaning the Homs vanish in both directions.

[KP17, Lemma 5.1] applies, hence f^* is fully faithful and $D^b(X)$ has the semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \dots, \mathcal{B}_X(M-1) \rangle, \quad (3.1)$$

where $\mathcal{B}_X := f^*\mathcal{B}$. We call $\mathcal{A}_X = \langle \mathcal{B}_X, \dots, \mathcal{B}_X(M-1) \rangle^\perp$ the *Kuznetsov component*.

For subsets $T \subset \mathbf{Z}$ and $S \subset \mathbf{Z}/n$, define

$$\mathcal{B}_X^S(T) := \langle \mathcal{B}_X(t) \otimes \rho_k \rangle_{t \in T, k \in S} \subset D^b(Y)^{\mu_n}.$$

Proposition 3.2. *We have the following semiorthogonal decomposition,*

$$D^b(X)^{\mu_n} = \langle \mathcal{A}_X^{\mu_n}, \mathcal{B}_X^{[0, n-1]}([0, M-1]) \rangle.$$

Proof. Each piece of the semiorthogonal decomposition (3.1) is preserved by the action of μ_n . Thus, by Theorem 2.8 the action of μ_n distributes through the semiorthogonal decomposition. Furthermore, since the action of μ_n is trivial on each piece $\mathcal{B}_X(i)$, we get $\mathcal{B}_X(i)^{\mu_n} = \mathcal{B}_X^{[0, n-1]}(i)$ by Proposition 2.10. The result follows. \square

We also state a useful lemma which we will use in the next section.

Lemma 3.3 ([KP17, Lemma 6.1]). *For any twist $t \in \mathbf{Z}$ and weight $k \in \mathbf{Z}/n$ we have*

$$\mathbf{R}_{j_{k*}D^b(Z)}(\mathcal{B}_X^k(t)) = \mathcal{B}_X^{k+1}(t-d).$$

3.2. A semiorthogonal decomposition of $\mathcal{A}_X^{\mu_n}$. For this section, assume moreover that $0 < M < d$. Before stating the main theorem, let us set up some more notation. Define the functor $\Phi_k: D^b(X)^{\mu_n} \rightarrow D^b(X)^{\mu_n}$ for $0 \leq k \leq n-2$ as

$$\Phi_k(-) := \mathbf{L}_{\mathcal{B}_X^{[0, k]}([0, M-1])}(- \otimes \rho_k).$$

Furthermore, define the subcategory \mathcal{A}_Z of $D^b(X)^{\mu_n}$ as follows:

$$\mathcal{A}_Z := \langle j_{0*}D^b(Z), \mathcal{B}_X^1([M-d, -1]) \rangle. \quad (3.2)$$

Theorem 3.4. *Let X and Z be as in Section 3.1, with $0 < M < d$. We have a semiorthogonal decomposition,*

$$\mathcal{A}_X^{\mu_n} = \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z) \rangle.$$

Remark 3.5. The results of [KP17] apply to the numerical range $M \geq d$, while we work with $0 < M < d$. See Section 3.3 for a comparison between the results.

Proof. We follow the strategy of the proof in Section 6 of [KP17]. From Theorem 3.1, we have the semiorthogonal decomposition

$$D^b(X)^{\mu_n} = \langle \mathcal{B}_X^0([0, m-1]), j_{0*}D^b(Z), \dots, j_{n-2*}D^b(Z) \rangle. \quad (3.3)$$

Now rewrite the first component of the semiorthogonal decomposition above as

$$\begin{aligned} \mathcal{B}_X^0([0, m-1]) &= \langle \mathcal{B}_X^0([0, M-1]), \mathcal{B}_X^0([M, M+d-1]), \\ &\quad \mathcal{B}_X^0([M+d, M+2d-1]), \dots, \mathcal{B}_X^0([m-d, m-1]) \rangle. \end{aligned} \quad (3.4)$$

We next substitute the decomposition (3.4) into the semiorthogonal decomposition (3.3), and iteratively apply right mutations together with Lemma 3.3. Then, as in [KP17], we get

$$\begin{aligned} D^b(X)^{\mu_n} &= \langle \mathcal{B}_X^0([0, M-1]), j_{0*}D^b(Z), \mathcal{B}_X^1([M-d, M-1]), j_{1*}D^b(Z), \\ &\quad \mathcal{B}_X^2([M-d, M-1]), \dots, j_{n-2*}D^b(Z), \mathcal{B}_X^{n-1}([M-d, M-1]) \rangle. \end{aligned} \quad (3.5)$$

Now we deviate from the proof in *loc. cit.* and use \mathcal{A}_Z as defined above. Since $M < d$,

$$\mathcal{B}_X^k([M-d, M-1]) = \langle \mathcal{B}_X^k([M-d, -1]), \mathcal{B}_X^k([0, M-1]) \rangle$$

for any weight k . Also, since $j: Z \rightarrow X$ is equivariant, $j_{l*}(-) \otimes \rho_k = j_{l+k*}$. Therefore, (3.5) reads:

$$\begin{aligned} D^b(X)^{\mu_n} = & \langle \mathcal{B}_X^0([0, M-1]), \\ & \mathcal{A}_Z, \mathcal{B}_X^1([0, M-1]), \\ & \mathcal{A}_Z \otimes \rho_1, \mathcal{B}_X^2([0, M-1]), \\ & \dots, \\ & \mathcal{A}_Z \otimes \rho_{n-2}, \mathcal{B}_X^{n-1}([0, M-1]) \rangle. \end{aligned} \quad (3.6)$$

We next apply left mutations and regroup as follows.

$$\begin{aligned} D^b(X)^{\mu_n} = & \langle \mathbf{L}_{\mathcal{B}_X^0([0, M-1])}(\mathcal{A}_Z), \\ & \mathbf{L}_{\mathcal{B}_X^0([0, M-1])} \mathbf{L}_{\mathcal{B}_X^1([0, M-1])}(\mathcal{A}_Z \otimes \rho_1), \\ & \dots, \\ & \mathbf{L}_{\mathcal{B}_X^0([0, M-1])} \mathbf{L}_{\mathcal{B}_X^1([0, M-1])} \dots \mathbf{L}_{\mathcal{B}_X^{n-2}([0, M-1])}(\mathcal{A}_Z \otimes \rho_{n-2}), \\ & \mathcal{B}_X^0([0, M-1]), \mathcal{B}_X^1([0, M-1]), \dots, \mathcal{B}_X^{n-1}([0, M-1]) \rangle \\ = & \langle \mathbf{L}_{\mathcal{B}_X^0([0, M-1])}(\mathcal{A}_Z), \\ & \mathbf{L}_{\mathcal{B}_X^{[0,1]}([0, M-1])}(\mathcal{A}_Z \otimes \rho_1), \dots, \\ & \mathbf{L}_{\mathcal{B}_X^{[0, n-2]}([0, M-1])}(\mathcal{A}_Z \otimes \rho_{n-2}), \mathcal{B}_X^{[0, n-1]}([0, M-1]) \rangle \\ = & \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z), \mathcal{B}_X^{[0, n-1]}([0, M-1]) \rangle. \end{aligned} \quad (3.7)$$

Since the right-hand sides of the semiorthogonal decomposition above and in Proposition 3.2 match up, the components to the left are equivalent. The statement of the theorem follows. \square

Corollary 3.6. *Under the action of $\widehat{\mu}_n \simeq \mathbf{Z}/n$ on $D^b(X)^{\mu_n}$, we have*

$$\mathcal{A}_X = \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z) \rangle^{\mathbf{Z}/n}.$$

Proof. By [Ela14, Theorem 4.2], $(\mathcal{A}_X^{\mu_n})^{\mathbf{Z}/n} \simeq \mathcal{A}_X$. The result then follows by Theorem 3.4. \square

Example 3.7. Theorem 3.4 applies to the following classes of examples.

- (1) The Veronese double cone Y_1 , defined as the double cover of $\mathbf{P}(1, 1, 1, 2)$ branched over $Z \in |\mathcal{O}_{\mathbf{P}(1,1,1,2)}(6)|$. This is a smooth prime Fano threefold of index 2 and degree 1. In this case, $\mathcal{A}_Z = \langle D^b(Z), D^b(\text{pt}) \rangle$.
- (2) The prime Fano threefold X_2 of index 1 and genus 2, which is realized as a double cover of \mathbf{P}^3 branched in a sextic. Then $\mathcal{A}_Z = \langle D^b(Z), D^b(\text{pt}), D^b(\text{pt}) \rangle$.

Example 3.8 (The family X_4 and smoothings of $\frac{1}{4}(1, 1)$). Consider a Fano threefold X obtained as the double cover $f: X \rightarrow Q$ of a quadric hypersurface Q in \mathbf{P}^4 branched over a section of $\mathcal{O}_Q(2)$. The derived category of Q admits a strong full exceptional collection $D^b(Q) = \langle S', \mathcal{O}_Q, \mathcal{O}_Q(1), \mathcal{O}_Q(2) \rangle$, where S' is a spinor bundle on Q [Kap88]. This decomposition is not rectangular. Hence, the present setting is not covered by the results of this section. Nevertheless, we can apply the same methods and compare $\mathcal{A}_X^{\mu_2}$ and $D^b(Z)$. Denote by $\mathcal{C} = \langle S', \mathcal{O}_Q, \mathcal{O}_Q(1) \rangle$. By Theorem 3.1, and then arguing as in Lemma 3.3, we obtain a decomposition

$$\begin{aligned} D^b(X)^{\mu_2} = & \langle f_0^* \mathcal{C}, \mathcal{O}_X(2) \otimes \rho_0, j_{0*} D^b(Z) \rangle \\ = & \langle f_0^* \mathcal{C}, j_{0*} D^b(Z), \mathcal{O}_X \otimes \rho_1 \rangle. \end{aligned} \quad (3.8)$$

On the other hand, $\mathcal{O}_X = f^*\mathcal{O}_Q$ is exceptional by Kodaira vanishing. Let $\mathcal{A}_X := \langle \mathcal{O}_X \rangle^\perp$, so $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X \rangle$. Proposition 3.2 then gives

$$D^b(X)^{\mu_2} = \langle \mathcal{A}_X^{\mu_2}, \mathcal{O}_X \otimes \rho_0, \mathcal{O}_X \otimes \rho_1 \rangle.$$

Comparing the two decompositions we obtain

$$\langle \mathcal{A}_X^{\mu_2}, \mathcal{O}_X \otimes \rho_0 \rangle = \langle f_0^* \mathcal{C}, j_{0*} D^b(Z) \rangle. \quad (3.9)$$

The above equality can be used to relate the K-theories of $\mathcal{A}_X^{\mu_2}$ and $D^b(Z)$ (see Remark 5.7).

There exists another family of double covers, i.e. double covers of $\mathbf{P}(1, 1, 1, 2)$ branched over a section Z of $\mathcal{O}_{\mathbf{P}(1,1,1,2)}(8)$, which can be equivalently regarded as quartic hypersurfaces of $\mathbf{P} := \mathbf{P}(1, 1, 1, 2, 4)$. Let X be one such hypersurface. If X is chosen to miss the points $[0 : 0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 0 : 1]$, then X has two singularities $\frac{1}{2}(1, 1, 1)$ in its intersection with the line $x_0 = x_1 = x_2 = 0$, and is smooth otherwise. Now consider the canonical stack \mathcal{X} associated to X : it has index $M = 1$ and by Theorem 3.4 there is a semiorthogonal decomposition

$$\mathcal{A}_{\mathcal{X}}^{\mu_2} = \langle D^b(Z), D^b(\text{pt}), D^b(\text{pt}), D^b(\text{pt}) \rangle.$$

We point out that the family of threefolds X_4 and that of hypersurfaces just constructed admit a common degeneration as follows. It is well known that the cone over a rational normal quartic curve, i.e. the plane $\mathbf{P}(1, 1, 4)$, has smoothings both to a Veronese surface and to a quadric surface [Ste03, 14, Example 4]. Then, a cone over $\mathbf{P}(1, 1, 4)$ may be deformed to a singular quadric threefold and to $\mathbf{P}(1, 1, 1, 2)$, which is a cone over a Veronese surface. Thus, X_4 may be degenerated to the double cover of a singular quadric, which in turn degenerates to a cone over $\mathbf{P}(1, 1, 4)$.

3.3. Comparison to [KP17]. The article [KP17] works under the assumption $M \geq d$. If the inequality is strict, $D^b(Z)$ admits a Kuznetsov component and a decomposition as follows. Let \mathcal{B}_Z be the essential image of the restriction along the inclusion $Z \subset Y$. Then

$$D^b(Z) = \langle \mathcal{A}_Z^{\text{KP}}, \mathcal{B}_Z, \dots, \mathcal{B}_Z(M - d - 1) \rangle, \quad (3.10)$$

where $\mathcal{A}_Z^{\text{KP}}$ denotes here the right orthogonal, i.e. the Kuznetsov component of $D^b(Z)$, in contrast with (3.2). If $M = d$, then $D^b(Z) = \mathcal{A}_Z^{\text{KP}}$ (see [KP17, Lemma 5.5]).

In [KP17, Theorem 1.1] the authors show that for $M \geq d$ the μ_n -equivariant Kuznetsov component, $\mathcal{A}_X^{\mu_n}$, has a semiorthogonal decomposition into $n - 1$ copies of $\mathcal{A}_Z^{\text{KP}}$.

Theorem 3.4 shows that, when $0 < M < d$, there exists a category \mathcal{A}_Z , defined as a subcategory of $D^b(X)^{\mu_n}$ (see (3.2)), and a similar decomposition of $\mathcal{A}_X^{\mu_n}$. Observe that \mathcal{A}_Z now contains $D^b(Z)$ as a semiorthogonal component.

Suppose Y is weighted projective space. Then Y has a rectangular Lefschetz decomposition with $\mathcal{B} = \langle \mathcal{O}_Y \rangle$, and $\omega_Y \simeq \mathcal{O}_Y(-m)$. By adjunction, $\omega_Z \simeq \mathcal{O}_Z(nd - m)$. Hence the sign of $nd - m = d - M$ determines whether Z is Fano, K-trivial, or canonically polarized. The first two cases correspond to $M \geq d$ and are those considered in [KP17]. In summary,

$$\begin{array}{ll} d \leq M & \xrightarrow{\text{[KP17, Lemma 5.5]}} \mathcal{A}_Z^{\text{KP}} \hookrightarrow D^b(Z), \\ 0 < M < d & \xrightarrow{\text{Section 3.2}} D^b(Z) \hookrightarrow \mathcal{A}_Z. \end{array}$$

4. EQUIVARIANT EQUIVALENCES AND HODGE ISOMETRIES

4.1. Cohomology of varieties and Hodge theory. Let X be a smooth projective variety over \mathbf{C} of dimension n . For all k , the singular cohomology group $H^k(X, \mathbf{Z})$ carries a Hodge

structure. The complexification $H^k(X, \mathbf{C})$ decomposes as

$$H^k(X, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$, for all $p + q = k \geq 0$. We have $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Singular cohomology has the structure of a commutative ring with cup product. The class of a hyperplane section H of X induces the operation $- \cup H: H^k(X, \mathbf{Z}) \rightarrow H^{k+2}(X, \mathbf{Z})$. Then, *primitive cohomology* is defined as

$$H_{\text{prim}}^{n-k}(X, \mathbf{Z}) := \ker(- \cup H^{k+1}: H^{n-k}(X, \mathbf{Z}) \rightarrow H^{n+k+2}(X, \mathbf{Z}))$$

for all k . The Hodge structure on $H^{n-k}(X, \mathbf{Z})$ restricts to one on $H_{\text{prim}}^{n-k}(X, \mathbf{Z})$ [Voi07, II.6-7].

4.2. Topological K-theory. Our main references for topological K-theory are [Wei97] and the papers [Bla16], [Per22, Section 5.1], and [HLP20, Section 2]. We collect the results needed for this work below.

There is a lax monoidal functor

$$K^{\text{top}}: \text{Cat}_{\mathbf{C}} \rightarrow \text{Sp}$$

from \mathbf{C} -linear categories to the ∞ -category of spectra, which satisfies the following properties.

Theorem 4.1 ([Bla16]).

- (1) If $\mathcal{C} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ is a \mathbf{C} -linear semiorthogonal decomposition, then there is an equivalence

$$K^{\text{top}}(\mathcal{C}) \simeq K^{\text{top}}(\mathcal{C}_1) \oplus \dots \oplus K^{\text{top}}(\mathcal{C}_m), \quad (4.1)$$

where the map $K^{\text{top}}(\mathcal{C}) \rightarrow K^{\text{top}}(\mathcal{C}_i)$ is induced by the projection functor to \mathcal{C}_i .

- (2) Let $K(\mathcal{C})$, $\text{HN}(\mathcal{C})$, and $\text{HP}(\mathcal{C})$ denote algebraic K-theory, negative cyclic homology, and periodic cyclic homology of \mathcal{C} respectively. There is a functorial commutative square

$$\begin{array}{ccc} K(\mathcal{C}) & \longrightarrow & \text{HN}(\mathcal{C}) \\ \downarrow & & \downarrow \\ K^{\text{top}}(\mathcal{C}) & \longrightarrow & \text{HP}(\mathcal{C}) \end{array} \quad (4.2)$$

- (3) If X is a proper complex variety, then there exists a functorial equivalence $K^{\text{top}}(\text{D}_{\text{perf}}(X)) \simeq K^{\text{top}}(X)$, the complex K-theory spectrum of X . Under this equivalence, the left vertical arrow in (4.2) recovers the usual inclusion of algebraic K-theory into topological K-theory, and the bottom arrow coincides with the usual Chern character under the identification of $\text{HP}(\text{D}_{\text{perf}}(X))$ with 2-periodic de Rham cohomology.

For $\mathcal{C} \in \text{Cat}_{\mathbf{C}}$ and t an integer, write $K_t^{\text{top}}(\mathcal{C}) = \pi_t(K^{\text{top}}(\mathcal{C}))$. If in addition the category \mathcal{C} is proper, then for each integer t there is a bilinear Euler pairing $\chi^{\text{top}}: K_t^{\text{top}}(\mathcal{C}) \otimes K_t^{\text{top}}(\mathcal{C}) \rightarrow \mathbf{Z}$ induced by the evaluation functor

$$\mathcal{H}\text{om}(-, -): \mathcal{C}^{\text{op}} \otimes_{\text{D}_{\text{perf}}(\text{Spec}(\mathbf{C}))} \mathcal{C} \rightarrow \text{D}_{\text{perf}}(\text{Spec}(\mathbf{C})),$$

where D_{perf} denotes the category of perfect complexes. The Euler pairing satisfies the following properties [Per22, Lemma 5.2]:

- The pairing is compatible with the inclusions of $K_t^{\text{top}}(\mathcal{C}_i)$, and it makes (4.1) into a semiorthogonal sum (i.e. $\chi^{\text{top}}(v_i, v_j) = 0$ for $v_i \in K_t^{\text{top}}(\mathcal{C}_i)$, $v_j \in K_t^{\text{top}}(\mathcal{C}_j)$, $i > j$);

- when restricted to $K_0(\mathcal{C})$ via the inclusion in (4.2), χ^{top} coincides with the usual Euler pairing

$$\chi(E, F) = \sum_{i \in \mathbf{Z}} (-1)^i \dim \text{Hom}_{\mathcal{C}}^i(E, F).$$

- (Riemann–Roch) If $\mathcal{C} = D_{\text{perf}}(X)$, with X a proper complex variety, then for $v, w \in K_t^{\text{top}}(\mathcal{C})$ we have $\chi^{\text{top}}(v, w) = p_*(v^\vee \otimes w) \in K_{2t}^{\text{top}}(\text{Spec}(\mathbf{C})) \simeq \mathbf{Z}$, with $p: X \rightarrow \text{Spec}(\mathbf{C})$ the structure morphism.

Now we suppose \mathcal{C} is an admissible subcategory of the derived category of a proper smooth DM quotient stack, to leverage results on the non-commutative Hodge-to-de Rham spectral sequence. We say that $K^{\text{top}}(\mathcal{C})$ carries a *pure Hodge structure* if every homotopy group $K_t^{\text{top}}(\mathcal{C})$ carries a pure Hodge structure of weight $-t$.

Proposition 4.2. *Let \mathcal{C} be an admissible subcategory of $D_{\text{perf}}(\mathcal{X})$, for $\mathcal{X} = [X/G]$ a proper smooth DM stack.*

- (1) $K^{\text{top}}(\mathcal{C})$ comes endowed with a canonical pure Hodge structure with graded pieces

$$\text{gr}^p(K_t^{\text{top}}(\mathcal{C})_{\mathbf{C}}) \simeq \text{HH}_{t+2p}(\mathcal{C}), \quad (4.3)$$

where $\text{HH}_t(\mathcal{C})$ denotes the t -th Hochschild homology group of \mathcal{C} .

- (2) For $\mathcal{C} = D_{\text{perf}}(\mathcal{X})$, the Chern character induces an isomorphism of Hodge structures

$$K_t^{\text{top}}(D_{\text{perf}}(\mathcal{X}))_{\mathbf{Q}} \simeq \bigoplus_{k \in \mathbf{Z}} H^{2k-t}(I_{\mathcal{X}}, \mathbf{Q})(k), \quad (4.4)$$

where $I_{\mathcal{X}}$ denotes the (underived) inertia stack, and the summands on the right-hand side are its de Rham cohomology groups.

- (3) If $\mathcal{X} = X$ is a smooth proper complex variety, the summands in (4.4) coincide with $H^{2k-t}(X, \mathbf{Q})(k)$, the k -th Tate twists of rational singular cohomology groups.

Proof. To prove these statements, we claim that the map $K^{\text{top}}(\mathcal{C})_{\mathbf{C}} \rightarrow \text{HP}(\mathcal{C})$ in (4.2) is an isomorphism (this is referred to as the *lattice property* for \mathcal{C}), and that the non-commutative Hodge-to-de Rham sequence degenerates for \mathcal{C} (we say that \mathcal{C} has the *degeneration property*). Granting the claims, the degeneration property gives a canonical filtration of $\text{HP}_n(\mathcal{C})$ with p -th graded piece $\text{HH}_{n+2p}(\mathcal{C})$, which together with the lattice property shows (1).

There are identifications $\text{HP}_t(D_{\text{perf}}(\mathcal{X})) \simeq \bigoplus_{k \in \mathbf{Z}} H^{2k-t}(I_{\mathcal{X}}, \mathbf{Q})$ by [HLP20, Proposition 2.13], and $\text{HP}_t(D_{\text{perf}}(X)) \simeq H^{2k-t}(X, \mathbf{Q})$ by [Wei97], under which the non-commutative Hodge filtration coincides with the usual Hodge filtrations. This proves parts (2) and (3).

To establish the claims, observe first that they are preserved under direct summands and arbitrary direct sums in the category of noncommutative motives (defined in [Tab08]). Indeed, this holds for the lattice property because of the functoriality of (4.2), and for the degeneration property by [HLP20, Lemma 1.22]. Therefore, it is sufficient to establish them in the case $\mathcal{C} = D_{\text{perf}}(\mathcal{X})$, for which they are proven, respectively, in [HLP20, Corollary 2.19] and [HLP20, Corollary 1.23]. \square

4.3. Equivalences of Fourier–Mukai type induce Hodge isometries. Consider smooth proper DM stacks that are global quotients, $[X_j/G_j]$, for $j = 1, 2$. Fix admissible subcategories $i_1: \mathcal{K}_1 \rightarrow D^b(X_1)^{G_1}$ and $i_2: \mathcal{K}_2 \rightarrow D^b(X_2)^{G_2}$, and let i_j^* denote the left adjoints to the inclusions for $j = 1, 2$. Recall that an equivalence $\phi: \mathcal{K}_1 \rightarrow \mathcal{K}_2$ is said to be of *Fourier–Mukai type* if the composition $\psi: D^b(X_1)^{G_1} \rightarrow D^b(X_2)^{G_2}$ given by

$$D^b(X_1)^{G_1} \xrightarrow{i_1^*} \mathcal{K}_1 \xrightarrow{\phi} \mathcal{K}_2 \xrightarrow{i_2} D^b(X_2)^{G_2} \quad (4.5)$$

is a Fourier–Mukai functor. Denote by (\mathcal{E}_j, e_j) a dg-enhancement for each $D^b(X_j)^{G_j}$, which in turn induce enhancements (\mathcal{F}_j, f_j) for \mathcal{K}_j . In this setting:

Lemma 4.3. *An equivalence $\phi: \mathcal{K}_1 \rightarrow \mathcal{K}_2$ of Fourier–Mukai type lifts to an equivalence of dg-enhancements $(\mathcal{F}_1, f_1) \rightarrow (\mathcal{F}_2, f_2)$.*

Proof. The Fourier–Mukai assumption means that the composition ψ in (4.5) is of Fourier–Mukai type, whence it lifts to a functor $\Psi: (\mathcal{E}_1, e_1) \rightarrow (\mathcal{E}_2, e_2)$ (this is [Toë07, Theorem 8.9] for schemes, and [Kün23, Corollary 3.7] for the equivariant version). Let i_j^{dg} and $i_j^{*\text{dg}}$ denote the dg-lifts of the inclusions and their left adjoint functors. Define $\Phi := i_2^{*\text{dg}} \circ \Psi \circ i_1^{\text{dg}}$. The functor $\Phi: (\mathcal{F}_1, f_1) \rightarrow (\mathcal{F}_2, f_2)$ lifts ϕ . In fact, taking cohomology, we have

$$H^0(\Phi) \simeq i_2^* \circ H^0(\Psi) \circ i_1 \simeq i_2^* \circ i_2 \circ \phi \circ i_1^* \circ i_1 \simeq \phi. \quad \square$$

Observe that Proposition 4.2 applies to each \mathcal{K}_j , and endows each $K^{\text{top}}(\mathcal{K}_j)$ with Hodge structures. Then we have the following Proposition.

Proposition 4.4. *A Fourier–Mukai type equivalence $\phi: \mathcal{K}_1 \rightarrow \mathcal{K}_2$ induces a Hodge isometry $K^{\text{top}}(\mathcal{K}_1) \simeq K^{\text{top}}(\mathcal{K}_2)$.*

Proof. By the functoriality of K^{top} , and since the Euler pairing is defined through a canonical evaluation map, we immediately get isometries $K^{\text{top}}(\mathcal{K}_1) \simeq K^{\text{top}}(\mathcal{K}_2)$.

Since ϕ is of Fourier–Mukai type, it admits a lift $\Phi: (\mathcal{F}_1, f_1) \rightarrow (\mathcal{F}_2, f_2)$ to the dg-enhancements by Lemma 4.3. The construction of the Hodge filtration on K^{top} only depends on the noncommutative Hodge-to-de Rham spectral sequence, which is a motivic invariant of the dg-enhancements. Then $K^{\text{top}}(\mathcal{K}_1) \simeq K^{\text{top}}(\mathcal{K}_2)$ is a Hodge isometry. \square

5. APPLICATION: WEIGHTED DOUBLE SOLIDS

In this section we focus on the three-dimensional case.

5.1. Setup. We collect here our assumptions and notation.

Definition 5.1. We say that X is a *weighted double solid* if X is a three-dimensional smooth DM stack equipped with a 2:1 map to a weighted projective space Y (regarded as a smooth stack), branched over a divisor $Z \in |\mathcal{O}_Y(2d)|$. We say that X is *prime* if it has Picard rank 1.

In the notation of 3.1, a weighted double solid satisfies $n = 2$ and $M = m - d$. Here, m is the sum of the weights of the coordinates of Y . We will sometimes need an additional generality assumption, which ensures that Z has Picard rank 1 (this holds for X prime and very general). We refer to X in this case as a *very general* weighted double solid.

Remark 5.2. A weighted double solid with $0 < M$ is Fano. Indeed, denote by $\mathcal{O}_X(1)$ the (ample) pull-back of $\mathcal{O}_Y(1)$. Then, by the Riemann–Hurwitz formula we have $K_X = f^*K_Y + \mathcal{O}_X(d) = \mathcal{O}_X(-m + d) = \mathcal{O}_X(-M)$, and $M > 0$ is the index of X . Similarly, $K_Z = \mathcal{O}_Z(-m + 2d)$, so Z is canonically polarized if $M < d$ and K -trivial if $M = d$.

Kollár’s [Kol19, Theorem 29] implies the following Lemma 5.3 even without the generality assumption on X, X' in all cases except quartic surfaces in \mathbf{P}^3 . For completeness, we include a short argument which also covers quartic surfaces.

Lemma 5.3. *Let X, X' be weighted double solids, with X very general. If there is an isomorphism $Z \simeq Z'$, then $X \simeq X'$.*

Proof. Denote the isomorphism of branch divisors $\phi: Z \simeq Z'$. Since X is very general, Z is Picard rank 1, and hence so is Z' . This means that $\phi^*(h') = h$ where h, h' are the ample generators of $\text{Pic}(Z), \text{Pic}(Z')$, respectively. Therefore, global sections of (multiples of) h and h' are identified, and hence the embeddings of Z, Z' into \mathbf{P}^3 or $\mathbf{P}(w_0, \dots, w_3)$ coincide up to a projective transformation. Hence, the covers $X \simeq X'$ are isomorphic. \square

In the rest of the paper, we will often assume that weighted double solids satisfy $0 < M \leq d$. This will imply that the results of Section 3 hold, including the limit case $\mathcal{A}_Z \simeq D^b(Z)$.

5.2. Hodge theory and Kuznetsov components. Here we use the results of Section 4 to relate equivalences of Kuznetsov components to the cohomology of the ramification divisors. Unless stated otherwise, we fix a weighted double solid X with $0 < M \leq d$. By Theorem 3.4, we have the following semiorthogonal decomposition of the μ_n -equivariant Kuznetsov component.

$$\mathcal{A}_X^{\mu_2} = \langle \mathbf{L}_{\mathbb{B}} j_{0*} D^b(Z), \mathcal{E}_1, \dots, \mathcal{E}_{d-M} \rangle \subset D^b(X)^{\mu_2},$$

where the \mathcal{E}_i are exceptional objects, so that $\langle \mathcal{E}_i \rangle \simeq D^b(\text{pt})$. By the discussion in Section 4.2, we have an isomorphism of Hodge structures

$$K_{-t}^{\text{top}}(\mathcal{A}_X^{\mu_2}) \simeq K_{-t}^{\text{top}}(D^b(Z)) \oplus \bigoplus_{i=1}^{d-M} K_{-t}^{\text{top}}(D^b(\text{pt})) \quad (5.1)$$

for all integers t .

If \mathcal{C} is a proper category, we will write

$$K_0(\mathcal{C})^\perp := \{e \in K_0^{\text{top}}(\mathcal{C}) \mid \chi^{\text{top}}(a, e) = 0 \text{ for all } a \in K_0(\mathcal{C})\}.$$

Recall that the Chern character induces a functorial isomorphism of Hodge structures $K_0^{\text{top}}(D^b(Z))_{\mathbb{Q}} \simeq H^{\text{even}}(Z, \mathbb{Q})$, by Proposition 4.2. We denote its image in $H^{\text{even}}(Z, \mathbb{Q})$ by Λ_Z . Observe that, since Z is a smooth surface, $\Lambda_Z \subset H^0(Z, \mathbb{Z}) \oplus H^2(Z, \mathbb{Z}) \oplus \frac{1}{2}H^4(Z, \mathbb{Z})$.

Lemma 5.4. *Suppose that X is a prime weighted double solid with $0 < M$. Then*

- (1) *there is an isometry $K_0(\mathcal{A}_X^{\mu_2})^\perp \simeq K_0(D^b(Z))^\perp$.*
- (2) *The Chern character map in Proposition 4.2 restricts to an isometry on $K_0(D^b(Z))^\perp$, and its image is contained in $H_{\text{prim}}^2(Z, \mathbb{Z}) \subset \Lambda_Z$.*

Proof. For (1), first suppose $0 < M \leq d$. Then, by definition of $K_0(\mathcal{A}_X^{\mu_2})^\perp$ and through the decomposition (5.1), a class $e \in K_0^{\text{top}}(\mathcal{A}_X^{\mu_2})$ lies in $K_0(\mathcal{A}_X^{\mu_2})^\perp$ if and only if $e \in K_0(D^b(Z))^\perp$ and $\chi^{\text{top}}([\mathcal{E}_i], e) = 0$ for all i (here we use that $[\mathcal{E}_i] \in K_0(\mathcal{A}_X^{\mu_2})$). Since (5.1) is semiorthogonal, this is equivalent to saying that $e \in K_0(D^b(Z))^\perp$. Arguing similarly, if $d < M$ the decomposition (3.10) implies (1).

To prove (2), we suppose $[E] \in K_0(D^b(Z))^\perp$. Let h be an ample divisor on Z (corresponding to a hyperplane section H), and fix a point $z \in Z$. Then $[\mathcal{O}_Z], [\mathcal{O}_h], [\mathcal{O}_z] \in K_0(D^b(Z))$, so $[E]$ is orthogonal to them. The topological Euler pairing on $D^b(Z)$ satisfies Hirzebruch–Riemann–Roch. Since $\dim Z = 2$, these orthogonality conditions become:

$$0 = \chi^{\text{top}}(\mathcal{O}_z, E) = \text{ch}_0(E), \quad (5.2)$$

$$0 = \chi^{\text{top}}(\mathcal{O}_h, E) = \int_Z \left(0, -h, -\frac{h^2}{2}\right) \cdot \text{ch}(E) \cdot \text{td}(Z) = -h \cdot \text{ch}_1(E) \quad (5.3)$$

$$0 = \chi_{\text{top}}(\mathcal{O}_Z, E) = \int_Z \text{ch}(E) \cdot \text{td}(Z) = \text{ch}_2(E) - \text{ch}_1(E) \cdot \text{td}_1(Z). \quad (5.4)$$

Moreover, $\text{td}_1(Z)$ is an algebraic class, i.e. $\text{td}_1(Z) = \sum_i a_i E_i$ where for each i , $a_i \in \mathbb{Q}$ and E_i is an effective divisor. Again, since $E \in K_0(\mathcal{A}_X^{\mu_2})^\perp$, we have:

$$0 = \chi^{\text{top}}(\mathcal{O}_{E_i}, E) = \int_Z \left(0, -E_i, -\frac{E_i^2}{2}\right) \cdot \text{ch}(E) \cdot \text{td}(Z) = -E_i \cdot \text{ch}_1(E). \quad (5.5)$$

Then by (5.4), $\text{ch}_2(E) = 0$. Together with (5.2) and (5.3), it follows that $\text{ch}(E) \in H_{\text{prim}}^2(Z, \mathbb{Z})$.

Finally, note that under the Chern character map, the Euler pairing $\chi^{\text{top}}(-, -)$ and the intersection pairing on $H^*(Z, \mathbf{Q})$ coincide for classes of the form $(0, \text{ch}_1, 0)$. Therefore, ch restricts to an isometry on $K_0(\text{D}^b(Z))^\perp$. \square

Proposition 5.5. *Suppose that X is a very general prime weighted double solid with $0 < M$. Then the Chern character induces an isometry*

$$K_0(\mathcal{A}_X^{\mu_2})^\perp \simeq H_{\text{prim}}^2(Z, \mathbf{Z}).$$

Proof. Suppose $[E] \in K_0^{\text{top}}(\text{D}^b(Z))$ satisfies $\text{ch}(E) \in H_{\text{prim}}^2(Z, \mathbf{Z})$. By Lemma 5.4, it is enough to show that $[E] \in K_0(\text{D}^b(Z))^\perp$. From the definition of $H_{\text{prim}}^2(Z, \mathbf{Z})$, we have

$$0 = \text{ch}_0(E) = h \cdot \text{ch}_1(E) = \text{ch}_2(E). \quad (5.6)$$

Since Z has Picard rank 1, we know

$$\text{td}(Z) = \left(1, -\frac{a}{2}h, \text{td}_2(Z)\right),$$

for some integer $a \in \mathbf{Z}$. In particular, $\text{ch}_1(E) \cdot \text{td}_1(Z) = 0$. As in the proof of Lemma 5.4, by Hirzebruch–Riemann–Roch it follows that

$$0 = \chi^{\text{top}}(\mathcal{O}_z, E) = \chi^{\text{top}}(\mathcal{O}_h, E) = \chi^{\text{top}}(\mathcal{O}_Z, E), \quad (5.7)$$

where $z \in Z$. By our assumptions on X , the Picard group $\text{Pic } Z$ has rank 1 and is generated by h . Hence $K_0(\text{D}^b(Z))$ has rank 3, and is generated by $[\mathcal{O}_Z], [\mathcal{O}_h], [\mathcal{O}_z]$. Therefore $\chi^{\text{top}}(a, E) = 0$ for all $a \in K_0(\text{D}^b(Z))$, so $[E] \in K_0(\text{D}^b(Z))^\perp$. \square

We now prove the main theorem of this section by combining the results above.

Theorem 5.6. *Let X and X' be prime weighted double solids covering the same weighted projective space, with X very general and $d = d'$. Suppose we have a Fourier–Mukai type equivalence $\Phi^{\mu_2}: \mathcal{A}_X^{\mu_2} \rightarrow \mathcal{A}_{X'}^{\mu_2}$. Then this induces a Hodge isometry*

$$H_{\text{prim}}^2(Z, \mathbf{Z}) \simeq H_{\text{prim}}^2(Z', \mathbf{Z}).$$

Proof. We first show that Φ^{μ_2} induces the required isomorphism of abelian groups preserving the bilinear pairings. By the functorial diagram (4.2), and since the Euler pairing is defined through a canonical evaluation map, we immediately get an isometry $\gamma: K_0(\mathcal{A}_X^{\mu_2})^\perp \simeq K_0(\mathcal{A}_{X'}^{\mu_2})^\perp$. This fits into the commutative square

$$\begin{array}{ccc} K_0(\mathcal{A}_X^{\mu_2})^\perp & \xrightarrow{\gamma} & K_0(\mathcal{A}_{X'}^{\mu_2})^\perp \\ \downarrow & & \downarrow \\ K_0^{\text{top}}(\mathcal{A}_X^{\mu_2}) & \xrightarrow{\delta} & K_0^{\text{top}}(\mathcal{A}_{X'}^{\mu_2}) \end{array}, \quad (5.8)$$

where δ is the Hodge isometry of Proposition 4.4, and the vertical maps are functorial inclusions.

Since X is very general, by Proposition 5.5 there is an isometry $\alpha: K_0(\mathcal{A}_X^{\mu_2})^\perp \simeq H_{\text{prim}}^2(Z, \mathbf{Z})$. Let α' be the composition of the maps in Lemma 5.4 applied to X' , i.e.

$$\alpha': K_0(\mathcal{A}_{X'}^{\mu_2})^\perp \xrightarrow{\sim} K_0(\text{D}^b(Z))^\perp \xrightarrow{\text{ch}} H_{\text{prim}}^2(Z', \mathbf{Z}).$$

This is an isometry onto its image. We claim that α' is in fact surjective. Consider the composition

$$\alpha' \circ \gamma \circ \alpha^{-1}: H_{\text{prim}}^2(Z, \mathbf{Z}) \xrightarrow{\sim} K_0(\mathcal{A}_X^{\mu_2})^\perp \xrightarrow{\sim} K_0(\mathcal{A}_{X'}^{\mu_2})^\perp \xrightarrow{\text{ch}} H_{\text{prim}}^2(Z', \mathbf{Z}). \quad (5.9)$$

Since Z and Z' are in the same linear series $|\mathcal{O}_Y(nd)|$, we have $b_2(Z) = b_2(Z')$. Hence $\text{rk}(H_{\text{prim}}^2(Z, \mathbf{Z})) = b_2(Z) - 1 = \text{rk}(H_{\text{prim}}^2(Z', \mathbf{Z}))$. Therefore α' is an isometric isomorphism, and so is (5.9).

There is another commutative square

$$\begin{array}{ccc} H_{\text{prim}}^2(Z, \mathbf{Z}) & \xrightarrow{\alpha^{-1}} & K_0(\mathcal{A}_X^{\mu_2})^\perp \\ \downarrow & & \downarrow \\ \Lambda_Z & \longrightarrow & K_0^{\text{top}}(\mathcal{A}_X^{\mu_2}) \end{array}, \quad (5.10)$$

where the vertical maps are the natural inclusions, and the bottom horizontal arrow is the inverse of the Chern character $\Lambda_Z \rightarrow K_0^{\text{top}}(Z)$ followed by the inclusion of the first factor of (5.1). The commutativity of the diagram implies that $K_0(\mathcal{A}_X^{\mu_2})^\perp$ carries a Hodge structure which, on the one hand, makes α into a morphism of Hodge structures, and on the other hand is induced by restricting the Hodge structure of $K_0^{\text{top}}(\mathcal{A}_X^{\mu_2})$. The same arguments show that α' is a morphism of Hodge structures. In particular, (5.8) is a commutative diagram of Hodge structures, where the horizontal arrows are isometries.

Therefore (5.9) is the desired isometry of Hodge structures $H_{\text{prim}}^2(Z, \mathbf{Z}) \simeq H_{\text{prim}}^2(Z', \mathbf{Z})$. \square

Remark 5.7. Alternatively, if $K_0(\mathcal{C})^\perp$ vanishes for $\mathcal{C} = {}^\perp\mathcal{A}_X^{\mu_2}$ and $\mathcal{C} = j_{0*}D^b(Z)^\perp$ (with orthogonals taken in $D^b(X)^{\mu_2}$), one can argue as above that $K_0(\mathcal{A}_X^{\mu_2})^\perp \simeq K_0(D^b(Z))^\perp$ by comparing the decompositions

$$\langle \mathcal{A}_X^{\mu_2}, {}^\perp\mathcal{A}_X^{\mu_2} \rangle = D^b(X)^{\mu_2} = \langle j_{0*}D^b(Z)^\perp, j_{0*}D^b(Z) \rangle.$$

given in Theorem 3.1 and Proposition 3.2. For example, one shows in this way that Proposition 5.5 and Theorem 5.6 hold for very general members of the family of Fano threefolds X_4 because of the decompositions (3.9).

Remark 5.8. It would be interesting to know whether the statement of Proposition 5.5 extends to higher dimensions, since then the argument of Theorem 5.6 would also carry through. We hope to explore this further in future work.

6. CATEGORICAL TORELLI THEOREMS

In this section, we apply Theorem 5.6 to prove a categorical Torelli theorem for the family of Fano threefolds X_2 (Theorem 6.4) and for the family of Fano threefolds Y_1 (Theorem 6.5). The latter was the only remaining index 2 open case.

Consider once again a weighted double solid X satisfying $0 < M$. Denote by $\tau: \mathcal{A}_X \rightarrow \mathcal{A}_X$ the categorical involution induced by the involution of the double cover X . Also define the *rotation functor* $R: \mathcal{A}_X \rightarrow \mathcal{A}_X$ by $R(-) := \mathbf{L}_{\mathcal{B}_X}(- \otimes \mathcal{O}_X(1))$. We have the following relationships between the Serre functor of \mathcal{A}_X , the rotation functor R , and the categorical involution τ :

Proposition 6.1. *There are isomorphisms of functors:*

- $R^d \simeq \tau[1]$,
- $S_{\mathcal{A}_X}^{-1} \simeq R^M[-m + 1]$.

Proof. The first bullet point is by [Kuz19, Corollary 3.18]. For the second bullet point, note that by Proposition 2.4 and Remark 5.2 we get

$$\begin{aligned} S_{\mathcal{A}_X}^{-1}(-) &= \mathbf{L}_{\mathbb{B}} \mathbf{L}_{\mathbb{B}(1)} \cdots \mathbf{L}_{\mathbb{B}(M-1)}(- \otimes \mathcal{O}_X(M))[-m+1] \\ &= \mathbf{R}\mathbf{L}_{\mathbb{B}} \mathbf{L}_{\mathbb{B}(1)} \cdots \mathbf{L}_{\mathbb{B}(M-2)}(- \otimes \mathcal{O}_X(M-1))[-m+1] \\ &\quad \cdots \\ &= \mathbf{R}^M[-m+1], \end{aligned}$$

by repeatedly using that $\mathbf{L}_F(- \otimes \mathcal{O}_X(1)) = \mathbf{L}_{F \otimes \mathcal{O}_X(-1)}(-) \otimes \mathcal{O}_X(1)$. \square

Let X' be another weighted double solid with the same values of m, d as X . We denote by $\tau \in \text{Aut}(\mathcal{A}_X)$ and $\tau' \in \text{Aut}(\mathcal{A}_{X'})$ the covering autoequivalences.

Lemma 6.2. *Suppose that an equivalence $\Phi: \mathcal{A}_X \simeq \mathcal{A}_{X'}$ commutes with the involutions, i.e. there is an isomorphism of functors $\Phi \circ \tau \simeq \tau' \circ \Phi$. Then Φ descends to an equivalence of equivariant categories $\Phi^{\mu_2}: \mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}$.*

Proof. All actions we discuss in this proof will be understood to be $\mathbf{Z}/2 \simeq \mu_2$ -actions. We first observe that Φ preserves 1-categorical actions (see Definition 2.5(1)). This is equivalent to the fact that Φ commutes with the involutions.

Next, we check that Φ intertwines the involutions, considered as 2-categorical actions on \mathcal{A}_X and $\mathcal{A}_{X'}$ (Definition 2.5(2)). The 1-categorical actions τ and τ' lift to 2-categorical actions because the functors $\tau: \mathcal{A}_X \rightarrow \mathcal{A}_X$ and $\tau': \mathcal{A}_{X'} \rightarrow \mathcal{A}_{X'}$ are given by pulling back geometric involutions and pullbacks are functorial. Moreover, these lifts are unique because $H^2(B\mathbf{Z}/2, \mathbf{C}^\times) = 0$ (see [BP23, Corollary 3.4] for the lifting criterion, and [BP23, Example 3.12] for the vanishing). Thus Φ sends the 2-categorical action τ to the unique 2-categorical action τ' . So Φ respects 2-categorical actions, as required. \square

Lemma 6.3. *Assume that M divides an odd multiple of d . Then an equivalence $\Phi: \mathcal{A}_X \simeq \mathcal{A}_{X'}$ commutes with the covering involutions.*

Proof. Write $d(2b+1) = Ma$ for some integers a, b . Using Proposition 6.1, up to shifts we have

$$\tau = \tau^{2b+1} \simeq \mathbf{R}^{d(2b+1)} = \mathbf{R}^{Ma} \simeq S_{\mathcal{A}_X}^{-a},$$

and $\tau' \simeq S_{\mathcal{A}_{X'}}^{-a}$, up to the same shift. Then, because Serre functors and shifts commute with equivalences of categories, Φ commutes with the involution. \square

6.1. The case of X_2 . We now consider the case where $Y = \mathbf{P}^3$. In this case, the constraint $0 < M = 4 - d \leq d$ leaves us with two families: sextic double solids for $d = 3$ (see Example 3.7(2)) and quartic double solids for $d = 2$ (see Section 6.3). We first prove a categorical Torelli theorem for sextic double solids X_2 .

Theorem 6.4. *Let X, X' be prime Fano threefolds of index 1, and genus 2, with X very general. Then an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ of Fourier–Mukai type implies $X \simeq X'$.*

Proof. In this case, $M = m - d = 1$ always divides d , so the equivalence Φ descends to the equivariant categories by 6.2 and 6.3.

Then, by Theorem 5.6 we get a Hodge isometry

$$H_{\text{prim}}^2(Z, \mathbf{Q}) \simeq H_{\text{prim}}^2(Z', \mathbf{Q}).$$

Next, we apply the Torelli theorem for generic hypersurfaces [Don83, p. 325] (see [Voi22, Theorem 0.2] for the statement in terms of a Hodge isometry), which implies $Z \simeq Z'$. Donagi's theorem applies to the present case, since its numerical assumptions are automatically satisfied as long as $d \neq 1, 2$. Now, by Lemma 5.3, we conclude that $X \simeq X'$. \square

6.2. **The case of Y_1 .** We now consider a double cover X of $\mathbf{P}(1, 1, 1, 2)$ branched in a sextic hypersurface Z . This cover is known as a *Veronese double cone* and is often denoted Y_1 in the literature (see Example 3.7(1)). These are prime Fano threefolds of index 2, and degree 1.

Theorem 6.5. *Let X, X' be Veronese double cones, with X very general. Suppose that we have an equivalence $\Phi: \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$ of Fourier–Mukai type which commutes with the covering involution. Then $X \simeq X'$.*

Proof. By assumption and Lemma 6.2, Φ descends to an equivariant equivalence $\mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}$. Then, by Theorem 5.6 we have a Hodge isometry $H_{\text{prim}}^2(Z, \mathbf{Q}) \simeq H_{\text{prim}}^2(Z', \mathbf{Q})$. Now by [Sai86, Theorem A], generic Torelli holds for the degree 6 hypersurface $Z \subset \mathbf{P}(1, 1, 1, 2)$. Thus $Z \simeq Z'$ and by Lemma 5.3 we get $X \simeq X'$. \square

Remark 6.6. The relations in Proposition 6.1 read $\mathbf{R}^3 \simeq \tau[1]$ and $\mathbf{S}_{\mathcal{A}_X}^{-1} \simeq \mathbf{R}^2[-3]$. Since any equivalence must commute with shifts and Serre functors, then Φ commutes with τ if and only if it commutes with \mathbf{R} .

6.3. **The case of Y_2 .** We now consider a double cover X of \mathbf{P}^3 branched in a quartic K3 $Z \subset \mathbf{P}^3$. This cover is known as the *quartic double solid* and is often referred to as Y_2 in the literature. Below, we give a new proof of a categorical Torelli theorem for quartic double solids. Other proofs have been obtained in [BT16, APR22, BP23, FLZ23].

Theorem 6.7. *Let X, X' be quartic double solids, with X very general. Then an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ implies $X \simeq X'$.*

Proof. Firstly, note that all equivalences in the quartic double solid case are of Fourier–Mukai type by [PLZ23, Theorem 1.3]. Also note that Lemma 6.3 holds in this case. Thus, as in the proof of Theorem 6.4, the equivalence implies a Hodge isometry $H_{\text{prim}}^2(Z, \mathbf{Q}) \simeq H_{\text{prim}}^2(Z', \mathbf{Q})$. Since X is very general, Z has Picard rank 1. This means that the primitive second cohomology of Z is Hodge isometric to the transcendental lattice of Z [Huy16, Lemma 3.3.1]. So the Hodge isometry on primitive cohomology gives an isometry between the transcendental lattices of Z and Z' . Therefore, by [Huy16, Corollary 16.3.7], Z and Z' are derived equivalent via a Fourier–Mukai functor. By [Ogu02, Theorem 1.7], Z can only have one trivial Fourier–Mukai partner, so $Z \simeq Z'$. Finally, by Lemma 5.3 we get $X \simeq X'$. \square

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