

INFINITESIMAL CATEGORICAL TORELLI THEOREMS FOR FANO THREEFOLDS

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ABSTRACT. Let X be a smooth Fano variety and $\mathcal{K}u(X)$ its Kuznetsov component. A Torelli theorem for $\mathcal{K}u(X)$ states that $\mathcal{K}u(X)$ is uniquely determined by a certain polarized abelian variety associated to it. An infinitesimal Torelli theorem for X states that the differential of the period map is injective. A categorical variant of the infinitesimal Torelli theorem for X states that the morphism $\eta : H^1(X, T_X) \rightarrow \mathrm{HH}^2(\mathcal{K}u(X))$ is injective. In the present article, we use the machinery of Hochschild (co)homology to relate the aforementioned three Torelli-type theorems for smooth Fano varieties via a commutative diagram. As an application, we prove infinitesimal categorical Torelli theorems for a class of prime Fano threefolds. We then prove, infinitesimally, a restatement of the Debarre–Iliev–Manivel conjecture regarding the general fiber of the period map for ordinary Gushel–Mukai threefolds.

1. INTRODUCTION

Torelli problems are some of the oldest and the most classical problems in various aspects of algebraic geometry, including Hodge theory, birational geometry, moduli spaces of algebraic varieties, etc. The classical Torelli question asks whether an algebraic variety X is uniquely determined by an abelian variety associated to it. Denote by \mathcal{P} the period map $\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}$, where \mathcal{M} is the moduli space of some class of algebraic varieties up to isomorphism, and \mathcal{D} is the period domain. A *Torelli theorem* holds for X if and only if \mathcal{P} is injective. An *infinitesimal Torelli theorem* holds for X if and only if the differential $d\mathcal{P}$ of the period map is injective. If X is a smooth Fano threefold of Picard rank 1 – the focus of the second part of our paper – then the period map \mathcal{P} is given by $X \mapsto J(X)$, where $J(X)$ is the intermediate Jacobian of X .

On the other hand, the seminal work [BO01] shows that the bounded derived category $D^b(X)$ of a smooth projective Fano variety determines the isomorphism class of X . In other words, a *derived Torelli theorem* holds for X . It is natural to ask for a class of Fano varieties whether they are also determined by *less* information than the whole derived category.

A natural candidate is a subcategory $\mathcal{K}u(X) \subset D^b(X)$ called the *Kuznetsov component*, which is defined as the semiorthogonal complement of a natural exceptional collection of vector bundles on X . It is widely believed that the Kuznetsov component encodes the essential geometric information of X , and it has been studied extensively by Kuznetsov and others (e.g. in [Kuz04, Kuz09a, KP18]) for many Fano varieties. In particular, for a smooth cubic threefold X , its Kuznetsov component $\mathcal{K}u(X)$ determines its isomorphism class (see [BMMS12] and [PY21]). This is known in the literature as a *categorical Torelli theorem*.

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As in the case of the classical Torelli problem, one could imagine that the association $X \mapsto \mathcal{K}u(X)$ is a “categorical period map” \mathcal{P}_{cat} lifting the classical period map, defined on the moduli space \mathcal{M} of smooth Fano threefolds up to isomorphism. But since there is no good notion of a moduli space of semiorthogonal components of $D^b(X)$, we cannot make sense of \mathcal{P}_{cat} mathematically. Nevertheless, its differential $\eta : H^1(X, T_X) \rightarrow \text{HH}^2(\mathcal{K}u(X))$ is well-defined. We say an *infinitesimal categorical Torelli theorem* holds for X if the map η is injective.

Recently, the intermediate Jacobian of a smooth Fano variety X was reconstructed from its Kuznetsov component $\mathcal{K}u(X)$ in [Per22]. This relates the Hodge-theoretic and categorical invariants of X , in one direction.

Example 1.1 (Remark 3.8). When X is a Fano threefold of index one or two, it is clear that $H^1(X, \mathbb{Z}) \cong \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}) = 0$ since X is simply connected. According to [Per22, Lemma 5.2, Proposition 5.23], there is a Hodge isometry $K_{-3}^{\text{top}}(\mathcal{K}u(X))_{\text{tf}} \cong H^3(X, \mathbb{Z})_{\text{tf}}$ which preserves both pairings. The left hand side pairing is the Euler pairing, and the right hand side is the cohomology pairing. Then, $J(\mathcal{K}u(X))$ is an abelian variety with a polarization induced from the Euler pairing. Moreover, we have an isomorphism of abelian varieties $J(\mathcal{K}u(X)) \cong J(X)$. If there is a Fourier–Mukai equivalence $\mathcal{K}u(X_1) \simeq \mathcal{K}u(X_2)$, where X_1 and X_2 are Fano threefolds of index one or two, then $J(X_1) \cong J(X_2)$.

1.1. Main Results.

1.1.1. *Infinitesimal Torelli vs. infinitesimal categorical Torelli.* In the present article, we relate infinitesimal Torelli theorems and infinitesimal categorical Torelli theorems for a class of Fano threefolds of Picard rank one, using the machinery of Hochschild (co)homology. We write

$$\text{H}\Omega_{\bullet}(X) = \bigoplus_{p-q=\bullet} H^p(X, \Omega_X^q)$$

and

$$\text{HT}^{\bullet}(X) = \bigoplus_{p+q=\bullet} H^p(X, \Lambda^q T_X).$$

We prove the following theorem:

Theorem 1.2 (Theorem 3.10). *Let X be a smooth projective variety. Assume there is a semiorthogonal decomposition $D^b(X) = \langle \mathcal{K}u(X), E_1, \dots, E_n \rangle$, where $\{E_1, \dots, E_n\}$ is an exceptional collection. Then we have a commutative diagram*

$$\begin{array}{ccc} \text{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma} & \text{Hom}(\text{H}\Omega_{-1}(X), \text{H}\Omega_1(X)) \\ \alpha' \uparrow & \nearrow \tau & \nearrow \\ \text{HT}^2(X) & & \\ \text{inclusion} \uparrow & \nearrow d\mathcal{P} & \\ H^1(X, T_X) & & \end{array}$$

where τ is defined as a contraction of polyvector fields.

Remark 1.3. The map $\eta := d\mathcal{P}_{\text{cat}} : H^1(X, T_X) \rightarrow \text{HH}^2(\mathcal{K}u(X))$ is defined as the composition of the vertical maps in the commutative diagram.

If we assume the vanishing of the higher odd degree Hochschild homology of X , the space of first-order deformations of $J(\mathcal{K}u(X))$ is given by $\mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X)))$ by Proposition 3.9. If we assume further that $J(\mathcal{K}u(X)) \cong J(X)$ as abelian varieties, then the diagram from the theorem above can be interpreted as taking tangent spaces of the diagram

$$\begin{array}{ccc} \{\mathcal{K}u(X)\}/\simeq & \longrightarrow & \text{Abelian Varieties}/\cong \\ \uparrow & \nearrow & \\ \{X\}/\cong & & \end{array}$$

if we have good knowledge of the moduli spaces in question.

Corollary 1.4. *Infinitesimal classical Torelli for X implies infinitesimal categorical Torelli for the Kuznetsov component $\mathcal{K}u(X)$.*

Proof. Suppose $d\mathcal{P}$ is injective. Then the fact that we have a composition $d\mathcal{P} = \gamma \circ \eta$ implies that η is injective too. \square

The main examples for the problems of infinitesimal categorical Torelli that we study are those of Picard rank one, index one and two Fano threefolds. Recall that Fano threefolds satisfy the assumption $\mathrm{HH}_{2i+1}(X) = 0$ for $i \geq 1$ and $J(\mathcal{K}u(X)) \cong J(X)$ as abelian varieties (see Example 1.1 or Remark 3.8). We summarise our results in the following theorem.

Theorem 1.5 (Theorems 4.4, 4.6, 4.10, 4.11). *Let X_{2g-2} be a Fano threefold of index one and degree $2g - 2$, where g is its genus. Let Y_d be a Fano threefold of index two and degree d .*

- *For Y_d where $1 \leq d \leq 4$, the infinitesimal categorical Torelli theorem holds.*
- *For X_{2g-2} where $g = 2, 4, 5, 7$, the infinitesimal categorical Torelli theorem holds and the same result holds for X_4 if it is not hyperelliptic.*

1.1.2. *The Debarre–Iliev–Manivel Conjecture.* In [DIM12], the authors conjecture that the general fiber of the classical period map from the moduli space of ordinary Gushel–Mukai (GM) threefolds to the moduli space of 10 dimensional principally polarised abelian varieties is the disjoint union of $\mathcal{C}_m(X)$ and $M_G^X(2, 1, 5)$, both quotiented by involutions. We call this the *Debarre–Iliev–Manivel Conjecture*.

Within the moduli space of smooth GM threefolds, we define the fiber of the “categorical period map” at $\mathcal{K}u(X)$ as the set of isomorphism classes of all ordinary GM threefolds X' whose Kuznetsov components satisfy $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$. In our recent work [JLLZ21], we prove the categorical analogue of the Debarre–Iliev–Manivel conjecture:

Theorem 1.6 ([JLLZ21, Theorem 1.7]). *A general fiber of the categorical period map at the Kuznetsov component $\mathcal{K}u(X)$ of an ordinary GM threefold X is the union of $\mathcal{C}_m(X)/\iota$ and $M_G^X(2, 1, 5)/\iota'$, where ι, ι' are geometrically meaningful involutions.*

As an application, the Debarre–Iliev–Manivel conjecture can be restated in an equivalent form as follows:

Conjecture 1.7. *Let X be a general ordinary GM threefold. Then the intermediate Jacobian $J(X)$ determines the Kuznetsov component $\mathcal{K}u(X)$.*

Although we are not able to prove the conjecture, we are able to show that an infinitesimal version of it holds.

Theorem 1.8 (Theorem 4.6). *Let X be an ordinary GM threefold. Then the map*

$$\gamma : \mathrm{HH}^2(\mathcal{K}u(X)) \longrightarrow \mathrm{Hom}(\mathrm{H}\Omega_{-1}(X), \mathrm{H}\Omega_1(X))$$

is injective.

1.2. Organization of the paper. In Section 2, we collect basic facts about semiorthogonal decompositions. In Section 3, we recall the definition of Hochschild (co)homology for admissible subcategories of bounded derived categories $D^b(X)$ of smooth projective varieties X . We then prove Theorem 1.2. In Section 4, we apply the techniques developed in Section 3 to prime Fano threefolds of index one and two. In particular, we show the infinitesimal version (Theorem 1.8) of Conjecture 1.7 for ordinary GM threefolds.

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2. SEMIORTHOGONAL DECOMPOSITIONS

In this section, we collect some useful facts/results about semiorthogonal decompositions. Background on triangulated categories and derived categories of coherent sheaves can be found in [Huy06], for example. From now on, let $D^b(X)$ denote the bounded derived category of coherent sheaves on a smooth projective variety X .

2.1. Exceptional collections and semiorthogonal decompositions.

Definition 2.1. Let \mathcal{D} be a triangulated category. We say that $E \in \mathcal{D}$ is an *exceptional object* if $\mathrm{RHom}(E, E) = k$. Now let (E_1, \dots, E_m) be a collection of exceptional objects in \mathcal{D} . We say it is an *exceptional collection* if $\mathrm{RHom}(E_i, E_j) = 0$ for $i > j$.

Definition 2.2. Let \mathcal{D} be a triangulated category and \mathcal{C} a triangulated subcategory. We define the *right orthogonal complement* of \mathcal{C} in \mathcal{D} as the full triangulated subcategory

$$\mathcal{C}^\perp = \{X \in \mathcal{D} \mid \mathrm{Hom}(Y, X) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The *left orthogonal complement* is defined similarly, as

$${}^\perp\mathcal{C} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X, Y) = 0 \text{ for all } Y \in \mathcal{C}\}.$$

Definition 2.3. Let \mathcal{D} be a triangulated category. We say a triangulated subcategory $\mathcal{C} \subset \mathcal{D}$ is *admissible*, if the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ has left adjoint i^* and right adjoint $i^!$.

Definition 2.4. Let \mathcal{D} be a triangulated category, and $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ be a collection of full admissible subcategories of \mathcal{D} . We say that $\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ is a *semiorthogonal decomposition* of \mathcal{D} if $\mathcal{C}_j \subset \mathcal{C}_i^\perp$ for all $i > j$, and the subcategories $(\mathcal{C}_1, \dots, \mathcal{C}_m)$ generate \mathcal{D} , i.e. the smallest strictly full triangulated subcategory of \mathcal{D} containing $\mathcal{C}_1, \dots, \mathcal{C}_m$ is equal to \mathcal{D} .

Definition 2.5. The *Serre functor* $S_{\mathcal{D}}$ of a triangulated category \mathcal{D} is the autoequivalence of \mathcal{D} such that there is a functorial isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \cong \mathrm{Hom}_{\mathcal{D}}(B, S_{\mathcal{D}}(A))^{\vee}$$

for any $A, B \in \mathcal{D}$.

Proposition 2.6. *Assume a triangulated category \mathcal{D} admits a Serre functor $S_{\mathcal{D}}$ and let $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be a semiorthogonal decomposition. Then $\mathcal{D} \simeq \langle S_{\mathcal{D}}(\mathcal{D}_2), \mathcal{D}_1 \rangle \simeq \langle \mathcal{D}_2, S_{\mathcal{D}}^{-1}(\mathcal{D}_1) \rangle$ are also semiorthogonal decompositions.*

Example 2.7. Let X be a smooth projective variety and $\mathcal{D} = \mathrm{D}^b(X)$. Then $S_X := S_{\mathcal{D}}(-) = (- \otimes \mathcal{O}(K_X))[\dim X]$.

3. HOCHSCHILD (CO)HOMOLOGY AND INFINITESIMAL TORELLI THEOREMS

3.1. Definitions. In this subsection, we recall some basics on Hochschild (co)homology of admissible subcategories of $\mathrm{D}^b(X)$, where X is a smooth projective variety. We refer the reader to [Kuz09b] for more details. For Hochschild (co)homology of dg-categories, we refer the reader to the paper [Kel98] and survey [Kel07].

Definition 3.1 ([Kuz09b]). Let X be a smooth projective variety, and \mathcal{A} be an admissible subcategory of $\mathrm{D}^b(X)$. Consider any semiorthogonal decomposition of $\mathrm{D}^b(X)$ that contains \mathcal{A} as a component. Let P be the kernel of the projection to \mathcal{A} . The *Hochschild cohomology* of \mathcal{A} is defined as

$$\mathrm{HH}^{\bullet}(\mathcal{A}) := \mathrm{Hom}^{\bullet}(P, P).$$

The *Hochschild homology* of \mathcal{A} is defined as

$$\mathrm{HH}_{\bullet}(\mathcal{A}) := \mathrm{Hom}^{\bullet}(P, P \circ S_X).$$

Lemma 3.2 ([Kuz09b, Theorem 4.5, Proposition 4.6]). *Let E be a strong compact generator of \mathcal{A} , and define $A = \mathrm{RHom}(E, E)$. Then there are isomorphisms*

$$\mathrm{HH}^{\bullet}(A) \cong \mathrm{HH}^{\bullet}(\mathcal{A}) \quad \text{and} \quad \mathrm{HH}_{\bullet}(A) \cong \mathrm{HH}_{\bullet}(\mathcal{A}).$$

Remark 3.3.

- (1) Let $\mathrm{Perf}_{\mathrm{dg}}(X)$ be a dg-enhancement of $\mathrm{Perf}(X)$ whose objects are K -injective perfect complexes, and let $\mathcal{A}_{\mathrm{dg}}$ be a dg-subcategory of $\mathrm{Perf}_{\mathrm{dg}}(X)$ whose objects are in \mathcal{A} . Then we have the isomorphisms

$$\mathrm{HH}_{\bullet}(\mathcal{A}) \cong \mathrm{HH}_{\bullet}(\mathcal{A}_{\mathrm{dg}}) \quad \text{and} \quad \mathrm{HH}^{\bullet}(\mathcal{A}) \cong \mathrm{HH}^{\bullet}(\mathcal{A}_{\mathrm{dg}})$$

because the morphism of dg-categories $\mathcal{A} \rightarrow \mathcal{A}_{\mathrm{dg}}$ is a derived Morita equivalence. Here \mathcal{A} is a dg-category with one object and the endomorphism of this unique object is the dg-algebra A ; the morphism sends the unique object to an K -injective resolution of E .

- (2) Since $\mathrm{Perf}(X)$ is a triangulated category with a unique enhancement [LO10], the Hochschild (co)homology of admissible subcategories of $\mathrm{Perf}(X)$ defined by Kuznetsov coincides with that of the subcategory of the dg-enhancement that naturally comes from the dg-enhancement of $\mathrm{Perf}(X)$.

Let A be a \mathbb{C} -algebra. Note that the Hochschild homology $\mathrm{HH}_\bullet(A)$ is a graded $\mathrm{HH}^\bullet(A)$ -module. The module structure is easily described by the definition of Hochschild (co)homology via the Ext and Tor functors. Consider a variety X . According to Hochschild–Kostant–Rosenberg (HKR) isomorphism, the degree 2 Hochschild cohomology has a factor $H^1(X, T_X)$ which is the first-order deformations of X . The action of $H^1(X, T_X)$ on the Hochschild homology via the module structure can be interpreted as deformations of a certain invariant with respect to the deformations of X . Here, we have the invariant $\mathrm{HH}_\bullet(X) \cong \bigoplus_{p-q=\bullet} H^p(X, \Omega_X^q)$ that is closely related to the intermediate Jacobian of X . When X is a Fano threefold, the action of $H^1(X, T_X)$ on $\mathrm{HH}_{-1}(X) \cong H^{2,1}(X)$ is the derivative of period map.

In the case of admissible subcategories of derived categories, we can describe the module structure by kernels.

Definition 3.4. Let \mathcal{A} be an admissible subcategory of $\mathrm{D}^b(X)$, and let P be the kernel of the left projection to \mathcal{A} . Take $\alpha \in \mathrm{HH}^a(\mathcal{A})$ and $\beta \in \mathrm{HH}_b(\mathcal{A})$. The action of α on β is the composition

$$P \xrightarrow{\beta} P \circ S_X[b] \xrightarrow{\alpha \otimes \mathrm{id}} P \circ S_X[a+b].$$

Proposition 3.5. *Let \mathcal{A} be an admissible subcategory of $\mathrm{Perf}(X)$. Let E be a strong compact generator of \mathcal{A} , and $A := \mathrm{RHom}(E, E)$. The isomorphisms $\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(\mathcal{A})$ and $\mathrm{HH}_\bullet(A) \cong \mathrm{HH}_\bullet(\mathcal{A})$ from Lemma 3.2 preserve both sides of the obvious module structure and algebra structure of Hochschild cohomology.*

Proof. We follow [Kuz09b, Theorem 4.5, Proposition 4.6]. There is an embedding

$$\mu : \mathrm{D}^b(A \otimes A^{\mathrm{opp}}) \rightarrow \mathrm{D}^b(X \times X)$$

such that $\mu(A) = P$. Thus $\mathrm{Hom}^\bullet(A, A) \cong \mathrm{Hom}^\bullet(P, P)$, which is compatible with the algebra structure since both of the algebra structures are defined by compositions. It remains to check the compatibility of the module structure. The composition

$$A \otimes_{A \otimes A^{\mathrm{opp}}} A \longrightarrow \mu(A) \otimes^{\mathbb{L}} \mu(A)^T \xrightarrow{\mathrm{R}\Gamma} \mathrm{R}\Gamma(\mu(A) \otimes^{\mathbb{L}} \mu(A)^T)$$

is a quasi-isomorphism by dévissage¹. Namely by using a semi-free resolution, it suffices to check the case $A \otimes A^{\mathrm{opp}}$. Since it is functorial, the quasi-isomorphism is compatible with the module structure.

Finally, we must check that the isomorphism $H^\bullet(X \times X, P \times P^T) \cong \mathrm{Hom}^\bullet(P, P \circ S_X)$ is compatible with the module structure. Firstly, we have a functorial isomorphism with respect to the factor P :

$$\mathrm{Hom}^\bullet(\mathcal{O}_{X \times X}, P \otimes \Delta_* \mathcal{O}_X) \cong \mathrm{Hom}^\bullet((\Delta_* \mathcal{O}_X)^\vee, P) \cong \mathrm{Hom}^\bullet((\Delta_* \mathcal{O}_X)^\vee \circ S_X, P \circ S_X).$$

By Grothendieck Duality,

$$\mathrm{Hom}^\bullet((\Delta_* \mathcal{O}_X)^\vee \circ S_X, P \circ S_X) \cong \mathrm{Hom}^\bullet(\Delta_* \mathcal{O}_X, P \circ S_X).$$

Thus we have the following functorial isomorphism with respect to P :

$$\mathrm{Hom}^\bullet(\mathcal{O}_{X \times X}, P \otimes \Delta_* \mathcal{O}_X) \cong \mathrm{Hom}^\bullet(\Delta_* \mathcal{O}_X, P \circ S_X).$$

Consider the triangle $P' \rightarrow \Delta_* \mathcal{O}_X \rightarrow P$ with respect to the semiorthogonal decomposition. It induces functorial morphisms with respect to the factor P , to $\mathrm{Hom}^\bullet(\mathcal{O}_{X \times X}, P \otimes P^T)$ and $\mathrm{Hom}^\bullet(P, P \circ S_X)$, respectively. Note that we have $(\Delta_* \mathcal{O}_X)^T = \Delta_* \mathcal{O}_X$. By vanishing results

¹ T is a transposition.

Definition 3.7 ([Per22, Definition 5.24]). Let \mathcal{A} be an admissible subcategory of $D^b(X)$ and consider the diagram

$$\begin{array}{ccccc} K_1^{\text{top}}(\mathcal{A}) & \xrightarrow{\text{ch}_1^{\text{top}}} & \text{HP}_1(\mathcal{A}) & \xrightarrow{\cong} & \bigoplus_n \text{HH}_{2n-1}(\mathcal{A}) \\ & \searrow^{P'} & & & \downarrow P \\ & & & & \text{HH}_1(\mathcal{A}) \oplus \text{HH}_3(\mathcal{A}) \oplus \cdots \end{array}$$

where P is the natural projection, and P' is the composition. Define the *intermediate Jacobian of \mathcal{A}* as

$$J(\mathcal{A}) = (\text{HH}_1(\mathcal{A}) \oplus \text{HH}_3(\mathcal{A}) \oplus \cdots) / \Gamma,$$

where Γ is the image of P' . Note that Γ is a lattice.

Remark 3.8. In general, $J(\mathcal{A})$ is a complex torus. When X is a Fano threefold of index one or two, we have a non-trivial admissible subcategory $\mathcal{K}u(X)$ called the Kuznetsov component (see the survey [Kuz16]). Clearly $H^1(X, \mathbb{Z}) = \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}) = 0$ since X is simply connected. According to [Per22, Lemma 5.2, Proposition 5.3], there is a Hodge isometry $K_{-3}^{\text{top}}(\mathcal{K}u(X))_{\text{tf}} \cong H^3(X, \mathbb{Z})_{\text{tf}}$ which preserves both pairings. The left hand side pairing is the Euler pairing, and the right hand side is the cohomology pairing. Then, $J(\mathcal{K}u(X))$ is an abelian variety with a polarization induced from the Euler pairing. Moreover, we have an isomorphism of abelian varieties $J(\mathcal{K}u(X)) \cong J(X)$. If there is an Fourier–Mukai equivalence $\mathcal{K}u(X_1) \simeq \mathcal{K}u(X_2)$ for Fano threefolds X_1 and X_2 , then $J(X_1) \cong J(X_2)$.

In the theorems below, we write

$$\text{H}\Omega_{\bullet}(X) = \bigoplus_{p-q=\bullet} H^p(X, \Omega_X^q)$$

and

$$\text{H}\Gamma^{\bullet}(X) = \bigoplus_{p+q=\bullet} H^p(X, \Lambda^q T_X).$$

Proposition 3.9. *Assume that there is a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{K}u(X), E_1, \dots, E_n \rangle$$

where $\{E_1, \dots, E_n\}$ is an exceptional collection. Also assume that $\text{HH}_{2n+1}(X) = 0$ for $n \geq 1$. Then the first-order deformation space of $J(\mathcal{K}u(X))$ is

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong \text{Hom}(\text{HH}_{-1}(\mathcal{K}u(X)), \text{HH}_1(\mathcal{K}u(X))).$$

Proof. Write $V := \text{HH}_1(\mathcal{K}u(X))$. Note that $V \cong \text{HH}_1(X)$ by the Additivity Theorem of Hochschild Homology. Since

$$\text{HH}_1(X) \cong \text{H}\Omega_1(X) = \overline{\text{H}\Omega}_{-1}(X) \cong \overline{\text{H}\text{H}}_{-1}(X) \cong \overline{\text{H}\text{H}}_{-1}(\mathcal{K}u(X)),$$

by twisted HKR, there is a conjugation $\overline{V} = \text{HH}_{-1}(\mathcal{K}u(X))$. Since the tangent bundle of a torus is trivial, we have

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong H^1(V/\Gamma, V \otimes \mathcal{O}_{V/\Gamma}) \cong V \otimes H^1(V/\Gamma, \mathcal{O}_{V/\Gamma}).$$

Since $H^1(V/\Gamma, \mathcal{O}_{V/\Gamma}) \cong \text{Hom}_{\text{anti-linear}}(V, k) \cong \text{Hom}_k(\overline{V}, k)$, we finally get the isomorphism

$$H^1(J(\mathcal{K}u(X)), T_{J(\mathcal{K}u(X))}) \cong \text{Hom}(\text{HH}_{-1}(\mathcal{K}u(X)), \text{HH}_1(\mathcal{K}u(X)))$$

as required. \square

When $\mathrm{HH}_{2n+1}(X) = 0$, $n \geq 1$, we define a linear map from the deformations of $\mathcal{K}u(X)$ to the first-order deformations of its intermediate Jacobian $J(\mathcal{K}u(X))$ by the action of cohomology:

$$\mathrm{HH}^2(\mathcal{K}u(X)) \longrightarrow \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X))).$$

This map can be interpreted as the derivative of the following map of “moduli spaces”

$$\{\mathcal{K}u(X)\}/\simeq \longrightarrow \{J(\mathcal{K}u(X))\}/\cong.$$

Theorem 3.10. *Let X be a smooth projective variety. Assume $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), E_1, E_2, \dots, E_n \rangle$ where $\{E_1, E_2, \dots, E_n\}$ is an exceptional collection. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma} & \mathrm{Hom}(\mathrm{H}\Omega_{-1}(X), \mathrm{H}\Omega_1(X)) \\ \alpha' \uparrow & \nearrow \tau & \\ \mathrm{HT}^2(X) & & \\ \text{inclusion} \uparrow & \nearrow & \\ H^1(X, T_X) & & \end{array}$$

where τ is defined as contraction of polyvector fields.

We break the proof into several lemmas. We write $\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{A} \rangle$ for the semiorthogonal decomposition, where $\mathcal{A} = \langle E_1, E_2, \dots, E_n \rangle$. Let P_1 be the kernel of the projection to $\mathcal{K}u(X)$, and P_2 the kernel of the projection to \mathcal{A} .

Lemma 3.11. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}^{t_1}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \times \mathrm{Hom}^{t_2}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) & \longrightarrow & \mathrm{Hom}^{t_1+t_2}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) . \\ \downarrow (\alpha, \beta) & & \downarrow \beta \\ \mathrm{Hom}^{t_1}(P_1, P_1) \times \mathrm{Hom}^{t_2}(P_1, P_1 \circ S_X) & \longrightarrow & \mathrm{Hom}^{t_1+t_2}(P_1, P_1 \circ S_X) \end{array}$$

The morphisms in the rows are the composition maps described in Definition 3.4.

Proof. Firstly we define the maps α and β . There are triangles

$$(1) \quad P_2 \longrightarrow \mathcal{O}_\Delta \longrightarrow P_1 \longrightarrow P_2[1],$$

$$(2) \quad P_2 \circ S_X \longrightarrow \mathcal{O}_\Delta \circ S_X \longrightarrow P_1 \circ S_X \longrightarrow P_2 \circ S_X[1]$$

Note that by a similar proof to [Kuz09b, Cor 3.10], we have $\mathrm{Hom}^\bullet(P_2, P_1) = 0$. Applying $\mathrm{Hom}(-, P_1)$ to the triangle (1), we get an isomorphism

$$\mathrm{Hom}^\bullet(\mathcal{O}_\Delta, P_1) \cong \mathrm{Hom}^\bullet(P_1, P_1).$$

Then applying $\mathrm{Hom}(\mathcal{O}_\Delta, -)$ to the triangle (1), we get a long exact sequence

$$(3) \quad \mathrm{Hom}^t(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \longrightarrow \mathrm{Hom}^t(\mathcal{O}_\Delta, P_1) \longrightarrow \mathrm{Hom}^{t+1}(\mathcal{O}_\Delta, P_2).$$

Since $\mathrm{Hom}^\bullet(\mathcal{O}_\Delta, P_1) \cong \mathrm{Hom}^\bullet(P_1, P_1)$, we get a long exact sequence

$$(4) \quad \mathrm{Hom}^t(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \xrightarrow{\alpha} \mathrm{Hom}^t(P_1, P_1) \longrightarrow \mathrm{Hom}^{t+1}(\mathcal{O}_\Delta, P_2).$$

Again, applying the functor $\mathrm{Hom}(-, P_1 \circ S_X)$ to the triangle (1), we obtain an isomorphism $\mathrm{Hom}^\bullet(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}^\bullet(P_1, P_1 \circ S_X)$ since $\mathrm{Hom}^\bullet(P_2, P_1 \circ S_X) = 0$ [Kuz09b, Cor 3.10]. Applying $\mathrm{Hom}(\mathcal{O}_\Delta, -)$ to triangle (2), we obtain a long exact sequence

$$(5) \quad \mathrm{Hom}^\bullet(\mathcal{O}_\Delta, P_2 \circ S_X) \longrightarrow \mathrm{Hom}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \xrightarrow{\beta} \mathrm{Hom}^\bullet(\mathcal{O}_\Delta, P_1 \circ S_X).$$

By the isomorphism $\mathrm{Hom}^\bullet(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}^\bullet(P_1, P_1 \circ S_X)$, we get a long exact sequence

$$(6) \quad \mathrm{Hom}^\bullet(\mathcal{O}_\Delta, P_2 \circ S_X) \longrightarrow \mathrm{Hom}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \xrightarrow{\beta} \mathrm{Hom}^\bullet(P_1, P_1 \circ S_X).$$

Secondly, we explain the commutative diagram. Take $t_1 = t_2 = 0$; the general cases are similar. Let $f \in \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ and $g \in \mathrm{Hom}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X)$. We denote the natural morphism $\mathcal{O}_\Delta \rightarrow P_1$ by L . Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{O}_\Delta & \xrightarrow{g} & \mathcal{O}_\Delta \circ S_X & \xrightarrow{f \otimes \mathrm{id}} & \mathcal{O}_\Delta \circ S_X \\ L \downarrow & & \downarrow L \otimes \mathrm{id} & & \downarrow L \otimes \mathrm{id} \\ P_1 & \xrightarrow{g'} & P_1 \circ S_X & \xrightarrow{f' \otimes \mathrm{id}} & P_1 \circ S_X \end{array}$$

The composition $(L \otimes \mathrm{id}) \circ g$ gives an element g' in $\mathrm{Hom}(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}(P_1, P_1 \circ S_X)$, that is $\beta(g) = g'$. Similarly, $\alpha(f) = f'$. By the uniqueness of the isomorphism $\mathrm{Hom}(\mathcal{O}_\Delta, P_1 \circ S_X) \cong \mathrm{Hom}(P_1, P_1 \circ S_X)$, we have

$$\beta((f \otimes \mathrm{id}) \circ g) = (f' \otimes \mathrm{id}) \circ g'.$$

□

Remark 3.12. The morphism β is an isomorphism if $t_1 = 2$ and $t_2 = -1$. Indeed, let $h \in \mathrm{Hom}^\bullet(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X \circ S_X)$. According to [Kuz09b, Lemma 5.3], there is a commutative diagram

$$\begin{array}{ccc} \Delta_* \mathcal{O}_X & \xrightarrow{L} & P_1 \\ h \downarrow & & \downarrow \gamma_{P_1}(h) \\ \Delta_* \mathcal{O}_X \circ S_X[\bullet] & \xrightarrow{L \otimes \mathrm{id}} & P_1 \circ S_X[\bullet] \end{array}$$

Hence, $\beta = \gamma_{P_1}$ (see [Kuz09b, Section 5] for the definition of γ_{P_1}). Therefore, by the Theorem of Additivity [Kuz09b, Theorem 7.3], β is an isomorphism when $t_1 = 2$ and $t_2 = -1$.

Lemma 3.13. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{HT}^2(X) \times \mathrm{H}\Omega_{-1}(X) & \xrightarrow{\gamma'} & \mathrm{H}\Omega_1(X) \\ \downarrow (\alpha', \mathrm{id}) & & \downarrow \mathrm{id} \\ \mathrm{HH}^2(\mathcal{K}u(X)) \times \mathrm{H}\Omega_{-1}(X) & \xrightarrow{\gamma} & \mathrm{H}\Omega_1(X) \end{array}$$

Here, the map α' is the composition of the maps

$$\mathrm{HT}^2(X) \xrightarrow{\mathrm{IK}^{-1}} \mathrm{Hom}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \xrightarrow{\alpha} \mathrm{Hom}^2(P_1, P_1) = \mathrm{HH}^2(\mathcal{K}u(X)).$$

The map γ' in the first row is the natural action of polyvectors on forms: when restricting to $H^1(X, T_X)$, it is exactly the derivative of the period map. The map γ is defined by the

cohomology action as follows. Let $w \in \text{Hom}^2(P_1, P_1)$. Then $\gamma(w): \text{H}\Omega_{-1}(X) \rightarrow \text{H}\Omega_1(X)$ is defined by the commutative diagram

$$\begin{array}{ccc}
 \text{Hom}^{-1}(P_1, P_1 \circ S_X) & \xrightarrow{w} & \text{Hom}^1(P_1, P_1 \circ S_X) \\
 \uparrow \beta & & \downarrow \beta^{-1} \\
 \text{Hom}^{-1}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) & & \text{Hom}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \\
 \uparrow \text{IK}^{-1} & & \downarrow \text{IK} \\
 \text{H}\Omega_{-1}(X) & \xrightarrow{\gamma(w)} & \text{H}\Omega_{-1}(X)
 \end{array}$$

Proof. We have a commutative diagram

$$\begin{array}{ccc}
 \text{HT}^2(X) \times \text{H}\Omega_{-1}(X) & \xrightarrow{\gamma'} & \text{H}\Omega_1(X) \\
 \text{IK} \uparrow & & \text{IK} \uparrow \\
 \text{Hom}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \times \text{Hom}^{-1}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) & \longrightarrow & \text{Hom}^1(\mathcal{O}_\Delta, \mathcal{O}_\Delta \circ S_X) \\
 \downarrow (\alpha, \beta) & & \downarrow \beta \\
 \text{Hom}^2(P_1, P_1) \times \text{Hom}^{-1}(P_1, P_1 \circ S_X) & \longrightarrow & \text{Hom}^1(P_1, P_1 \circ S_X)
 \end{array}$$

The upper square is commutative by [CRVdB12, Theorem 1.4]. The lower square is commutative by Lemma 3.11. Therefore Lemma 3.13 follows by definitions. \square

Proof of Theorem 3.10. Thus, according to Lemma 3.13, we obtain a commutative diagram where η is defined to be the composition $\alpha' \circ$ (inclusion):

$$\begin{array}{ccc}
 \text{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma} & \text{Hom}(\text{H}\Omega_{-1}(X), \text{H}\Omega_1(X)) \\
 \alpha' \uparrow & \nearrow & \nearrow \\
 \text{HT}^2(X) & & \\
 \text{inclusion} \uparrow & \nearrow d\mathcal{P} & \\
 H^1(X, T_X) & &
 \end{array}$$

As a result, Theorem 3.10 is proved. \square

Corollary 3.14. *Let X be Fano threefolds of index one or two. Note here that we have $\text{H}\Omega_{-1}(X) = H^{2,1}(X)$ and $\text{H}\Omega_1(X) = H^{1,2}(X)$ ². Then there is a commutative diagram*

$$\begin{array}{ccc}
 \text{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma} & \text{Hom}(H^{2,1}(X), H^{1,2}(X)) \\
 \eta \uparrow & \nearrow d\mathcal{P} & \\
 H^1(X, T_X) & &
 \end{array}$$

²We have $h^{3,0} = h^{0,3} = 0$ for Fano threefolds

Remark 3.15. The commutative diagram above can be regarded as the infinitesimal version of the following “imaginary” maps

$$\begin{array}{ccc} \{\mathcal{K}u(X)\}/\cong & \longrightarrow & \{J(X)\}/\cong \\ \uparrow & \nearrow & \\ \{X\}/\cong & & \end{array}$$

Definition 3.16. Let X be a smooth projective variety. Assume there is a semiorthogonal decomposition $D^b(X) = \langle \mathcal{K}u(X), E_1, E_2, \dots, E_n \rangle$ where $\{E_1, E_2, \dots, E_n\}$ is an exceptional collection.

- (1) The variety X satisfies *infinitesimal Torelli* if

$$d\mathcal{P} : H^1(X, T_X) \longrightarrow \mathrm{Hom}(\mathrm{H}\Omega_{-1}(X), \mathrm{H}\Omega_1(X))$$

is injective;

- (2) The variety X satisfies *infinitesimal categorical Torelli* if the composition

$$\eta : H^1(X, T_X) \longrightarrow \mathrm{HH}^2(\mathcal{K}u(X))$$

is injective;

- (3) The Kuznetsov component $\mathcal{K}u(X)$ satisfies *infinitesimal Torelli* if

$$\gamma : \mathrm{HH}^2(\mathcal{K}u(X)) \longrightarrow \mathrm{Hom}(\mathrm{H}\Omega_{-1}(X), \mathrm{H}\Omega_1(X))$$

is injective.

Remark 3.17. The definitions (2) and (3) from Definition 3.16 depend on the choice of the equivalence class of Kuznetsov component $\mathcal{K}u(X)$.

4. INFINITESIMAL (CATEGORICAL) TORELLI THEOREMS FOR FANO THREEFOLDS OF INDEX ONE AND TWO

In this section, we apply Corollary 3.14 to establish infinitesimal categorical Torelli theorems for a class of prime Fano threefolds of index one and two, via the classical infinitesimal Torelli theorems for them.

We briefly recall the classification of Fano threefolds into deformation families via their Picard rank, index, and degree (or equivalently genus). The Picard rank of a Fano threefold X is the rank of $\mathrm{Pic} X$. In our paper, we will only be concerned with Picard rank one Fano threefolds, i.e. when $\mathrm{Pic} X = \mathbb{Z}$. The index of X is the positive integer i_X such that $K_X = -i_X H$, where H is the generator of $\mathrm{Pic} X$. If X is a Fano threefold, then $1 \leq i_X \leq 4$, and in this paper we will be concerned with the index one and two cases.

There are five deformation classes of Fano threefolds of index two, denoted Y_d . They are classified by their degree d , which takes the values $1 \leq d \leq 5$.

There are ten deformation classes of Fano threefolds of index one, denoted X_{2g-2} . They are classified by their degree (equivalently genus) $d = 2g - 2$, where g is the genus. The allowed values of g are $2 \leq g \leq 12$ and $g \neq 11$.

For a more detailed account of the Fano threefolds discussed above, see for example [Kuz04, Section 2].

4.1. Classical Torelli and infinitesimal Torelli theorems. Recall that the Torelli problem asks whether the period map $\mathcal{P} : \mathcal{M} \rightarrow \mathcal{D}$ is injective, while the infinitesimal Torelli problem asks whether the period map has injective differential. Let X be a smooth projective variety of dimension n over the complex numbers \mathbb{C} . We say that an *infinitesimal Torelli theorem* holds for X if the map

$$d\mathcal{P} : H^1(X, T_X) \longrightarrow \bigoplus_{p+q=n} \mathrm{Hom}(H^p(X, \Omega_X^q), H^{p+1}(X, \Omega_X^{q-1}))$$

is injective. Besides the case of curves, the injectivity of the above map has been studied for many other varieties:

- hypersurfaces in (weighted) projective spaces ([CGGH83, Don83, Sai86];
- complete intersections in projective spaces ([Ter90, Pet76, Usu76];
- zero loci of sections of vector bundles ([Fle86];
- certain cyclic covers of a Hirzebruch surfaces ([Kon85]);
- complete intersections in certain homogeneous Kähler manifolds ([Kon86]);
- weighted complete intersections ([Usu78];
- quasi-smooth Fano weighted hypersurfaces ([FRZ19];
- index one prime Fano threefolds of degree 4 ([Lic22]).

In particular, if X is a smooth prime Fano threefold of index one or two, then the map $d\mathcal{P}$ becomes

$$d\mathcal{P} : H^1(X, T_X) \longrightarrow \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)),$$

since $h^{3,0}(X) = h^{0,3}(X) = 0$. Then the definition of *infinitesimal Torelli* for X coincides with Definition 3.16. We have the following proposition:

Proposition 4.1 ([Sai86, Fle86]).

- (1) Let X be a prime Fano threefold of index one. Then the map $d\mathcal{P}$ is injective if $\deg(X) \in \{2, 4, 6, 8\}$ and X is not hyperelliptic.
- (2) Let X be a prime Fano threefold of index two. Then the map $d\mathcal{P}$ is injective if $\deg(X) \in \{1, 2, 3, 4\}$.³
- (3) If X is an index one prime Fano threefold such that $\deg(X) \in \{10, 14, 18, 22\}$, or if $\deg(X) = 4$ and X is hyperelliptic, then $d\mathcal{P}$ is not injective.

4.2. Infinitesimal categorical Torelli theorems for Fano threefolds. In the following sections, we study the commutative diagram constructed in Corollary 3.14, and investigate whether the infinitesimal categorical Torelli theorem defined in Definition 3.16 holds for various Fano threefolds. We use the definition of the Kuznetsov component of a Fano threefold of index one or two from the survey paper [Kuz16] and we refer to [BFT21] and [Bel21] for the dimensions of $H^1(X, T_X)$ and $H^1(X, \Omega_X^2)$.

Let X be a smooth projective variety of dimension n . Assume $D^b(X) = \langle \mathcal{K}u(X), E, \mathcal{O}_X \rangle$ where E^\vee is a globally generated rank r vector bundle with vanishing higher cohomology. Let $\mathcal{N}_{X/\mathrm{Gr}}^\vee$ be the shifted cone lying in the triangle

$$\mathcal{N}_{X/\mathrm{Gr}}^\vee \longrightarrow \phi^* \Omega_{\mathrm{Gr}(r,V)} \longrightarrow \Omega_X,$$

where $V := H^0(X, E^\vee)$.

³If $\deg(X) = 5$, then $H^1(X, T_X) = 0$ so we exclude this case.

Theorem 4.2 ([Kuz09b, Theorem 8.8]). *Let the notation be as in the paragraph above. Then we have*

(1)

$$\begin{aligned} \cdots \longrightarrow \bigoplus_{p=0}^{n-1} H^{t-p}(X, \Lambda^p T_X) &\longrightarrow \mathrm{HH}^t(\langle E, \mathcal{O}_X \rangle^\perp) \longrightarrow \\ &\longrightarrow H^{t-n+2}(X, E^\perp \otimes E \otimes \omega_X^{-1}) \xrightarrow{\alpha} \bigoplus_{p=0}^{n-1} H^{t+1-p}(X, \Lambda^p T_X) \longrightarrow \cdots \end{aligned}$$

(2)

$$\begin{aligned} \cdots \longrightarrow \bigoplus_{p=0}^{n-2} H^{t-p}(X, \Lambda^p T_X) &\longrightarrow \mathrm{HH}^t(\langle E, \mathcal{O}_X \rangle^\perp) \longrightarrow \\ &\longrightarrow H^{t-n+2}(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee \otimes \omega_X^{-1}) \xrightarrow{\nu} \bigoplus_{p=0}^{n-2} H^{t+1-p}(X, \Lambda^p T_X) \longrightarrow \cdots \end{aligned}$$

(3) *If E is a line bundle, then $\nu = 0$ and*

$$\mathrm{HH}^t(\langle E, \mathcal{O}_X \rangle^\perp) \cong \bigoplus_{p=0}^{n-2} H^{t-p}(X, \Lambda^p T_X) \oplus H^{t-n+2}(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee \otimes \omega_X^{-1}).$$

4.3. Infinitesimal categorical Torelli for Fano threefolds of index two. An application of part (3) of Theorem 4.2 to the case of index two Fano threefolds of degree d , i.e. when $\mathcal{K}u(Y_d) = \langle \mathcal{O}_{Y_d}(-H), \mathcal{O}_{Y_d} \rangle^\perp$, gives the following result:

Theorem 4.3 ([Kuz09b, Theorem 8.9]). *The second Hochschild cohomology of the Kuznetsov component of an index two Fano threefold of degree d is given by*

$$\mathrm{HH}^2(\mathcal{K}u(Y_d)) = \begin{cases} 0, & d = 5 \\ k^3, & d = 4 \\ k^{10}, & d = 3 \\ k^{20}, & d = 2 \\ k^{35}, & d = 1. \end{cases}$$

Theorem 4.4. *Let Y_d be index two Fano threefolds of degree $1 \leq d \leq 4$. The commutative diagrams of Corollary 3.14 for Y_d are as follows:*

(1) Y_4 : η is an isomorphism, γ is injective, and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^3 & \xrightarrow{\gamma} & k^4 \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^3 & & \end{array}$$

(2) Y_3 : η is an isomorphism, γ is injective, and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^{10} & \xrightarrow{\gamma} & k^{25} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{10} & & \end{array}$$

(3) Y_2 : η is injective, and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^{20} & \xrightarrow{\gamma} & k^{100} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{19} & & \end{array}$$

(4) Y_1 : η is injective, and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^{35} & \xrightarrow{\gamma} & k^{441} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{34} & & \end{array}$$

Proof. The map $d\mathcal{P}$ is injective for $d = 1, 2, 3, 4$ by Proposition 4.1. Then according to Corollary 1.4, η is injective. Thus for the cases $d = 3, 4$, the map η is an isomorphism, hence γ is injective. \square

4.4. Infinitesimal categorical Torelli for Fano threefolds of index one. Gushel–Mukai (GM) threefolds are the index one Fano threefolds of genus 6 (equivalently, degree 10). They fall into two classes: ordinary and special. We study Corollary 3.14 for each of these cases in Section 4.4.1. Finally, in Sections 4.4.2 and 4.4.3, we study Corollary 3.14 for the remaining index one Fano threefolds.

Proposition 4.5. *The second Hochschild cohomology of the Kuznetsov component of an index one Fano threefold of genus $g \geq 6$ is given by*

$$\mathrm{HH}^2(\mathcal{K}u(X_{2g-2})) = \begin{cases} k^3, & g = 10 \\ k^6, & g = 9 \\ k^{10}, & g = 8 \\ k^{18}, & g = 7 \\ k^{20}, & g = 6. \end{cases}$$

Proof. In the cases $g = 8$ and 10, this follows from the equivalences $\mathcal{K}u(X_{2g-2}) \simeq \mathcal{K}u(Y_d)$ (see [Kuz09a]) and Theorem 4.3. For the cases $g = 7$ and 9, note that we always have $\mathcal{K}u(X) \simeq \mathrm{D}^b(C)$ for some curve C . The semiorthogonal decompositions are

$$\mathrm{D}^b(X_{16}) = \langle \mathrm{D}^b(C_3), \mathcal{E}_3, \mathcal{O}_{X_{16}} \rangle$$

$$\mathrm{D}^b(X_{12}) = \langle \mathrm{D}^b(C_7), \mathcal{E}_5, \mathcal{O}_{X_{12}} \rangle$$

where C_i is a curve of genus i and \mathcal{E}_r is a vector bundle of rank r . By the HKR isomorphism, $\mathrm{HH}^2(C) \cong H^2(C, \mathcal{O}_C) \oplus H^1(C, T_C) \oplus H^0(C, \Lambda^2 T_C) = H^1(C, T_C)$. The second equality follows from the fact that C is of dimension 1. The dimensions follow from the fact that the dimension of the moduli of curves of genus i is $3i - 3$. For the $g = 6$ case, the second Hochschild cohomology is computed in [KP18, Proposition 2.12]. \square

4.4.1. *The Gushel–Mukai case.* Let X be a GM threefold, which is either a quadric section of a linear section of codimension two of the Grassmannian $\mathrm{Gr}(2, 5)$, or a double cover of a degree 5 index two Fano threefold Y_5 ramified in a quadric hypersurface. The unique rank two stable vector bundle \mathcal{E} on X induces a morphism $\phi : X \rightarrow \mathrm{Gr}(2, 5)$ such that $\mathcal{E} \cong \phi^*T$, where T is the tautological rank 2 bundle on $\mathrm{Gr}(2, 5)$. There is a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle,$$

Theorem 4.6. *Let X be a GM threefold. The commutative diagram in Corollary 3.14 is as follows:*

$$\begin{array}{ccc} k^{20} & \xrightarrow{\gamma} & k^{100} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{22} & & \end{array}$$

Moreover, γ is injective and η is surjective. In particular, the Kuznetsov component $\mathcal{K}u(X)$ of an ordinary GM threefold satisfies infinitesimal Torelli.

Proof. The moduli stack \mathcal{X} of Fano threefolds of index one and degree 10 is a 22 dimensional smooth irreducible algebraic stack by [DIM12, p. 13]. By Section 7 of [DIM12] we have that the differential

$$d\mathcal{P} : H^1(X, T_X) \longrightarrow \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

of the period map $\mathcal{P} : \mathcal{X} \rightarrow \mathcal{A}_{10}$ has 2-dimensional kernel. Consider the kernel of η . Clearly, $\ker \eta \subset \ker d\mathcal{P}$ hence $\dim \ker \eta \leq 2$. Since the dimension of the image of η is less than or equal to 20, the dimension of $\ker \eta$ must be 2 and η is surjective. Finally, since image of $d\mathcal{P}$ is 20-dimensional, so is γ , hence γ is injective. \square

Remark 4.7. There is another proof for the case of ordinary GM threefolds X , which uses the long exact sequence in part (2) of Theorem 4.2:

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee(H)) \longrightarrow H^1(X, T_X) \xrightarrow{\eta} \mathrm{HH}^2(\mathcal{K}u(X)) \xrightarrow{\nu} \\ \xrightarrow{\nu} H^1(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee(H)) \longrightarrow H^2(X, T_X) \longrightarrow 0. \end{aligned}$$

It suffices to compute $H^0(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee(1))$ and $H^1(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee(1))$. Since $\mathcal{N}_{X/\mathrm{Gr}}$ is the restriction of $\mathcal{O}_{\mathrm{Gr}}(1) \oplus \mathcal{O}_{\mathrm{Gr}}(1) \oplus \mathcal{O}_{\mathrm{Gr}}(2)$, we have that $\mathcal{N}_{X/\mathrm{Gr}}^\vee(1)$ is the restriction of $\mathcal{O}_{\mathrm{Gr}} \oplus \mathcal{O}_{\mathrm{Gr}} \oplus \mathcal{O}_{\mathrm{Gr}}(-1)$, which is $\mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{O}_X(-H)$. Then $H^0(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee(H)) = k^2$ and $H^1(X, \mathcal{N}_{X/\mathrm{Gr}}^\vee(H)) = 0$ by the Kodaira Vanishing Theorem. Thus, η is surjective with 2-dimensional kernel, hence γ is injective.

4.4.2. *The cases of X_{18} , X_{16} , X_{14} , and X_{12} .* Consider index one prime Fano threefolds of genus $7 \leq g \leq 10$. They are

- (1) X_{12} , $g = 7$: a linear section of a connected component of the orthogonal Lagrangian Grassmannian $\mathrm{OGr}_+(5, 10) \subset \mathbb{P}^{15}$;
- (2) X_{14} , $g = 8$: a linear section of $\mathrm{Gr}(2, 6) \subset \mathbb{P}^{14}$;
- (3) X_{16} , $g = 9$: a linear section of the Lagrangian Grassmannian $\mathrm{LGr}(3, 6) \subset \mathbb{P}^{13}$;
- (4) X_{18} , $g = 10$: a linear section of the homogeneous space $G_2/P \subset \mathbb{P}^{13}$;

Theorem 4.8. *Let X and Y be smooth projective varieties. Suppose there is an equivalence of their Kuznetsov components $\mathcal{K}u(X) \simeq \mathcal{K}u(Y)$ which is induced by a Fourier–Mukai functor. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma_X} & \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(X)), \mathrm{HH}_1(\mathcal{K}u(X))) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^2(\mathcal{K}u(Y)) & \xrightarrow{\gamma_Y} & \mathrm{Hom}(\mathrm{HH}_{-1}(\mathcal{K}u(Y)), \mathrm{HH}_1(\mathcal{K}u(Y))) \end{array}$$

Proof. According to Theorem 3.6, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X)) \times \mathrm{HH}_{-1}(\mathcal{K}u(X)) & \xrightarrow{\gamma_X} & \mathrm{HH}_1(\mathcal{K}u(X)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^2(\mathcal{K}u(Y)) \times \mathrm{HH}_{-1}(\mathcal{K}u(Y)) & \xrightarrow{\gamma_Y} & \mathrm{HH}_1(\mathcal{K}u(Y)) \end{array}$$

The maps in the rows are defined as the cohomology action on homology. Hence the commutative diagram in the theorem follows. \square

Remark 4.9. When X and Y are Fano threefolds of index one and two, respectively, we have $\mathrm{HH}_{-1}(X) \cong H^{2,1}(X)$ and $\mathrm{HH}_{-1}(Y) \cong H^{2,1}(Y)$, respectively. Hence we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma_X} & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^2(\mathcal{K}u(Y)) & \xrightarrow{\gamma_Y} & \mathrm{Hom}(H^{2,1}(Y), H^{1,2}(Y)) \end{array}$$

where γ_X and γ_Y are the maps constructed in Theorem 3.14.

Theorem 4.10. *The diagrams in Corollary 3.14 for X_{18} , X_{16} , X_{14} , and X_{12} are as follows:*

(1) X_{18} : γ is injective.

$$\begin{array}{ccc} k^3 & \xrightarrow{\gamma} & k^4 \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{10} & & \end{array}$$

(2) X_{16} : γ is injective.

$$\begin{array}{ccc} k^6 & \xrightarrow{\gamma} & k^9 \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{12} & & \end{array}$$

(3) X_{14} : γ is injective.

$$\begin{array}{ccc} k^{10} & \xrightarrow{\gamma} & k^{25} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{15} & & \end{array}$$

(4) X_{12} : γ is injective, $d\mathcal{P}$ is injective, and η is an isomorphism.

$$\begin{array}{ccc} k^{18} & \xrightarrow{\gamma} & k^{49} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{18} & & \end{array}$$

Proof. In the cases of X_{18} , X_{16} , and X_{12} we always have $\mathcal{K}u(X) \simeq D^b(C)$ for some curve C . The semiorthogonal decompositions are

$$D^b(X_{18}) = \langle D^b(C_2), \mathcal{E}_2, \mathcal{O}_{X_{18}} \rangle$$

$$D^b(X_{16}) = \langle D^b(C_3), \mathcal{E}_3, \mathcal{O}_{X_{16}} \rangle$$

$$D^b(X_{12}) = \langle D^b(C_7), \mathcal{E}_5, \mathcal{O}_{X_{12}} \rangle$$

where C_i is a curve of genus i and \mathcal{E}_r is a vector bundle of rank r . We write X for X_{18} , X_{16} , and X_{12} . By the HKR isomorphism, $\mathrm{HH}^2(C) \cong H^2(C, \mathcal{O}_C) \oplus H^1(C, T_C) \oplus H^0(C, \wedge^2 T_C) = H^1(C, T_C)$. Note that we always refer to the version of HKR twisted by IK, as the ‘‘twisted HKR’’. This IK isomorphism preserves the module structure, where the geometric side is the action of polyvector fields on differential forms. Thus by Theorem 4.8 there is a commutative diagram

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma} & \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)) \\ \downarrow \cong & & \downarrow \cong \\ H^1(C, T_C) & \xrightarrow{d\mathcal{P}_C} & \mathrm{Hom}(H^{1,0}(C), H^{0,1}(C)) \end{array}$$

Therefore γ is injective for each X since $d\mathcal{P}_C$ is injective for each $C := C_i$. Indeed, C_3 is a plane quartic curve ([BF09, Section 3.1]), which is a canonical curve in \mathbb{P}^2 . Similarly, C_7 is also a canonical curve in \mathbb{P}^6 by [IM07, Section 1]. Thus they are both non-hyperelliptic.

For the case X_{14} , it is known that $\mathcal{K}u(X_{14}) \simeq \mathcal{K}u(Y_3)$ by [Kuz09a] (so also their Hochschild cohomologies are the same). Then by Theorem 4.8 there is a commutative diagram

$$\begin{array}{ccc} \mathrm{HH}^2(\mathcal{K}u(X_{14})) & \xrightarrow{\gamma_{X_{14}}} & \mathrm{Hom}(H^{2,1}(X_{14}), H^{1,2}(X_{14})) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HH}^2(\mathcal{K}u(Y_3)) & \xrightarrow{\gamma_{Y_3}} & \mathrm{Hom}(H^{2,1}(Y_3), H^{1,2}(Y_3)) \end{array}$$

Then $\gamma_{X_{14}}$ is injective since γ_{Y_3} is injective, by Theorem 4.4. \square

4.4.3. *The cases of X_8 , X_6 , non hyperelliptic X_4 , and X_2 .* Consider the index one prime Fano threefolds of genus $2 \leq g \leq 5$. They are

- (1) X_2 , $g = 2$: a double cover of \mathbb{P}^3 branched in a surface of degree six;
- (2) X_4 , $g = 3$: either a quartic threefold, or the double cover of a smooth quadric threefold branched in an intersection with a quartic;
- (3) X_6 , $g = 4$: a complete intersection of a cubic and a quadric;
- (4) X_8 , $g = 5$: a complete intersection of three quadrics.

In these cases, the Kuznetsov components are defined as $\langle \mathcal{O}_X \rangle^\perp$ by [BLMS17, Definition 6.5].

Theorem 4.11. *The diagrams in Corollary 3.14 for X_8 , X_6 , X_4 , and X_2 are as follows:*

(1) X_8 : γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^{27} & \xrightarrow{\gamma} & k^{196} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{27} & & \end{array}$$

(2) X_6 : γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^{34} & \xrightarrow{\gamma} & k^{400} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{34} & & \end{array}$$

(3) (a) If X_4 is a smooth quartic threefold, then γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

(b) If X_4 is a hyperelliptic Fano threefold, then η is an isomorphism and neither of γ and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^{45} & \xrightarrow{\gamma} & k^{900} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{45} & & \end{array}$$

(4) X_2 : γ is injective, η is an isomorphism, and $d\mathcal{P}$ is injective.

$$\begin{array}{ccc} k^{68} & \xrightarrow{\gamma} & k^{2704} \\ \eta \uparrow & \nearrow d\mathcal{P} & \\ k^{68} & & \end{array}$$

Proof. First, we prove that η is an isomorphism in each case. We write X for X_8 , X_6 , X_4 , and X_2 . Note that $D^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X \rangle$. Denote by P_1 the kernel of the left projection to $\mathcal{K}u(X)$, and P_2 the kernel of the right projection to $\langle \mathcal{O}_X \rangle$. There is a triangle

$$P_2 \rightarrow \Delta_* \mathcal{O}_X \rightarrow P_1 \rightarrow P_2[1].$$

Applying the functor $\Delta^!$ to the triangle, we obtain the diagram

$$\begin{array}{ccccc} \Delta^! P_2 & \longrightarrow & \Delta^! \Delta_* \mathcal{O}_X & \xrightarrow{L} & \Delta^! P_1 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \text{id} \\ \omega_X^{-1}[-3] & \xrightarrow{w} & \bigoplus_{p=0}^3 \Lambda^p T_X[-p] & \longrightarrow & \Delta^! P_1 \end{array}$$

According to [Kuz09b, Theorem 8.5], the map w is an isomorphism onto the third summand. Applying $\mathrm{Hom}^2(\mathcal{O}_X, -)$, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}^2(\mathcal{O}_X, \Delta^! \Delta_* \mathcal{O}_X) & \xrightarrow{L} & \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! P_1) \\
\downarrow \cong & & \downarrow \mathrm{id} \\
\mathrm{Hom}^2(\mathcal{O}_X, \bigoplus_{p=0}^3 \Lambda^p T_X[-p]) & \longrightarrow & \mathrm{Hom}^2(\mathcal{O}_X, \Delta^! P_1) \\
\cong \uparrow & \nearrow \cong & \\
\mathrm{Hom}^2(\mathcal{O}_X, \bigoplus_{p=0}^2 \Lambda^p T_X[-p]) & &
\end{array}$$

Thus, the morphism L is an isomorphism. However, L is naturally isomorphic to the morphism

$$\mathrm{Hom}^2(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X) \rightarrow \mathrm{Hom}^2(\Delta_* \mathcal{O}_X, P_1) \cong \mathrm{Hom}^2(P_1, P_2).$$

That is to say the map $\alpha' : \mathrm{HH}^2(X) \rightarrow \mathrm{HH}^2(\mathcal{K}u(X))$ constructed in Theorem 3.10 is an isomorphism. According to [BFT21, Appendix A], $H^0(X, \Lambda^2 T_X) = 0$ and it is clear that $H^2(X, \mathcal{O}_X) = 0$, hence η is an isomorphism.

The map $d\mathcal{P}$ is injective for X_8 , X_6 , non-hyperelliptic X_4 , and X_2 by Proposition 4.1. Then γ is injective for these cases because η is an isomorphism. The map $d\mathcal{P}$ is not injective for hyperelliptic X_4 , thus γ is not injective. \square

Remark 4.12. In the literature [Kuz21, Example 3.17], one can define an alternative version of the Kuznetsov component of X_6 as $\mathcal{K}u(X) := \langle \mathcal{U}_1, \mathcal{O}_X \rangle^\perp$, where \mathcal{U}_1 is restriction of one of the spinor bundles on the quadric $M \subset \mathbb{P}^5$. Then we still have the commutative diagram from Corollary 3.14 as follows:

$$\begin{array}{ccc}
\mathrm{HH}^2(\mathcal{K}u(X)) & \xrightarrow{\gamma} & k^{400} \\
\eta \uparrow & \nearrow d\mathcal{P} & \\
k^{34} & &
\end{array}$$

Since $d\mathcal{P}$ is injective, η is injective. Thus infinitesimal categorical Torelli holds for X_6 . By Lemma 4.13, the second Hochschild cohomology is given by $\mathrm{HH}^2(\mathcal{K}u(X)) \cong k^{34}$, thus η is an isomorphism, hence γ is injective.

Lemma 4.13. *Let X be an index one prime Fano threefold of genus 4 with semiorthogonal decomposition*

$$\mathrm{D}^b(X) = \langle \mathcal{K}u(X), \mathcal{U}_1, \mathcal{O}_X \rangle.$$

Then $\mathrm{HH}^2(\mathcal{K}u(X)) \cong H^1(X, T_X) \cong k^{34}$.

Proof. Consider the tautological short exact sequence

$$0 \rightarrow \mathcal{U}_1 \rightarrow \mathcal{O}_X^{\oplus 4} \rightarrow \mathcal{Q}_1 \rightarrow 0,$$

where \mathcal{Q}_1 the tautological quotient bundle. It is known that $\mathcal{Q}_1 \cong \mathcal{U}_2^\vee$ by [Ott88, Theorem 2.8(ii)]. It is easy to see that $\mathcal{U}_1^\perp \cong \mathcal{Q}_1^\vee \cong \mathcal{U}_2$. By Theorem 4.2, there is a long exact sequence

$$\cdots \rightarrow H^0(X, \mathcal{U}_1^\perp \otimes \mathcal{U}_1^\vee) \rightarrow H^1(X, T_X) \rightarrow \mathrm{HH}^2(\mathcal{K}u(X)) \rightarrow H^1(X, \mathcal{U}_1^\perp \otimes \mathcal{U}_1^\vee) \rightarrow \cdots.$$

Note that $H^\bullet(X, \mathcal{U}_1^\perp \otimes \mathcal{U}_1^\vee) \cong \mathrm{Hom}^\bullet(\mathcal{U}_1, \mathcal{U}_2) = 0$ since $\mathcal{U}_1, \mathcal{U}_2$ are completely orthogonal by [Kuz21, Remark 5.19]. Hence $\mathrm{HH}^2(\mathcal{K}u(X)) \cong H^1(X, T_X) \cong k^{34}$. \square

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